

دوسية الزيود:

تفاضل و تكامل 3

calculus 3

سيف الزيود

إعداد

اللجنة الأكاديمية لقسم الهندسة الصناعية

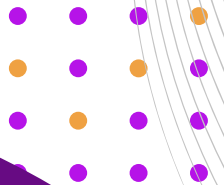
2025



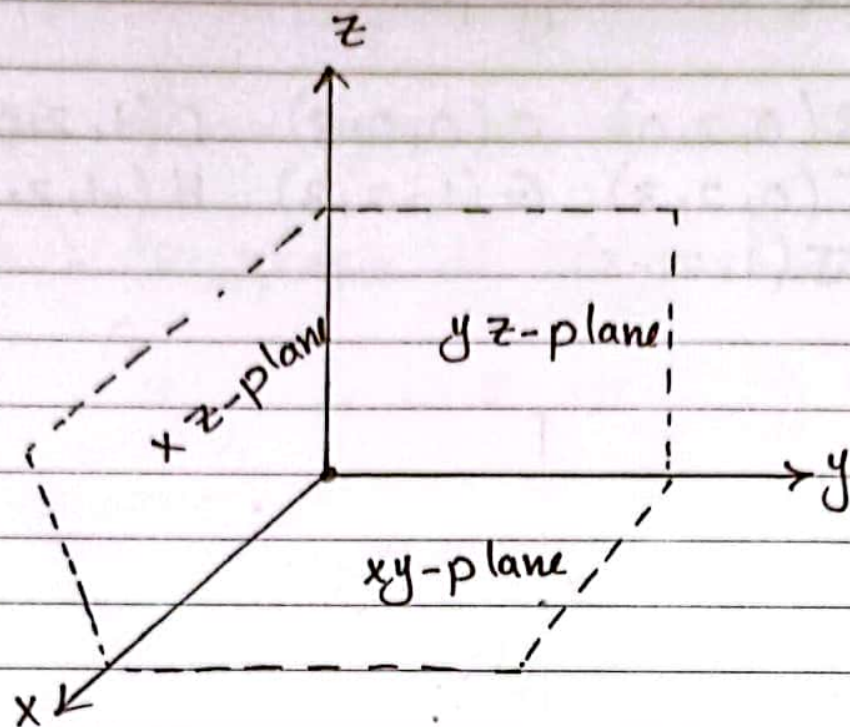
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12.1) Three-Dimensional Coordinate Systems



These axis are called the coordinate axis

The coordinate planes divide the space into 8 parts, each part is called an octant

The point $P(a, b, c)$

a : x-coordinate / b : y-coordinate / c : z-coordinate

The set of all numbers is $\mathbb{R} = (-\infty, \infty)$

The cartesian product $\mathbb{R} \times \mathbb{R} = (x, y)$ is called (2-D)

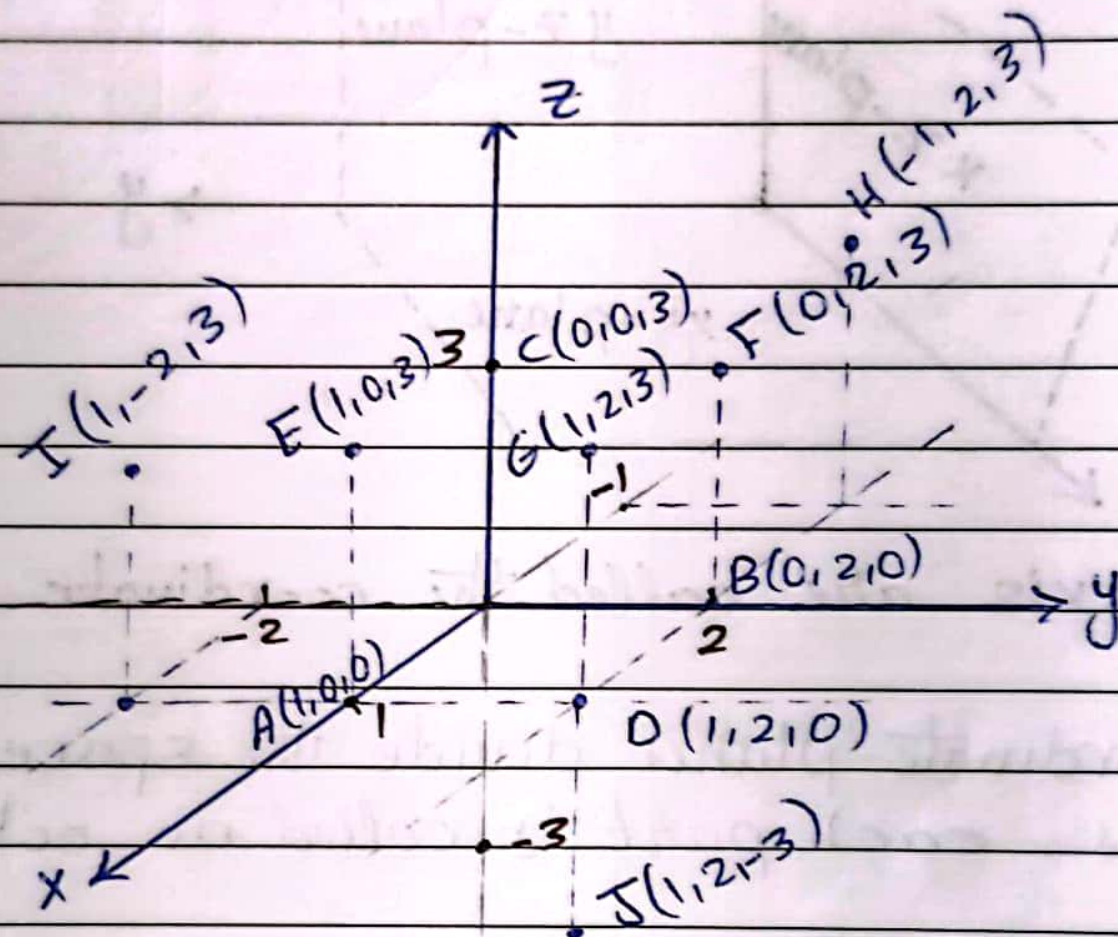
The cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = (x, y, z)$ is called (3-D)

Ex : Plote the Following points in the space

A(1,0,0) B(0,2,0) C(0,0,3) D(1,2,0)

E(1,0,3) F(0,2,3) G(1,2,3) H(-1,2,3)

I(1,-2,3) J(1,2,-3)

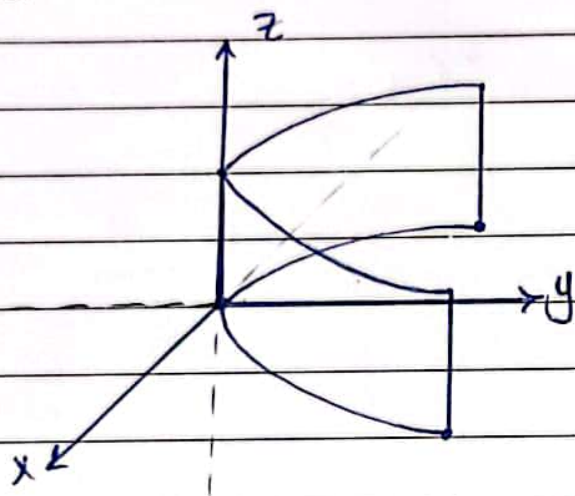


The graph of an equation in 2D is a curve.

The graph of an equation in 3D is a surface.

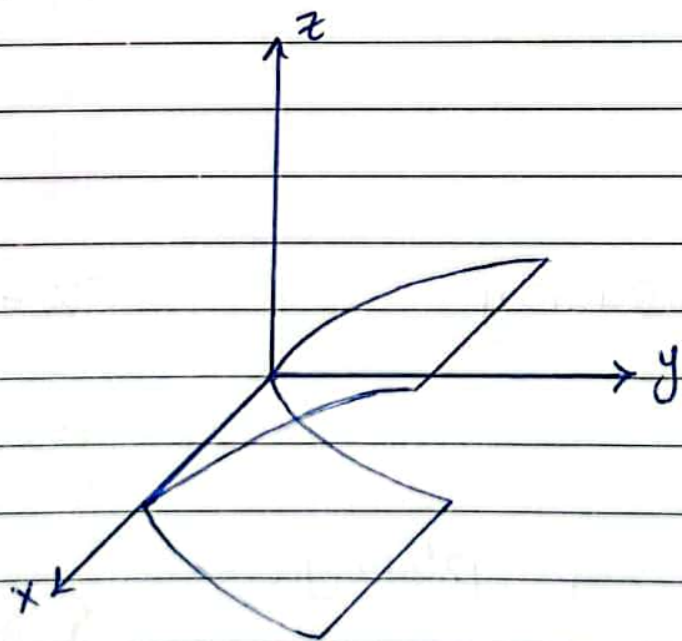
Ex: sketch the graph of the surface :

① $y = x^2$ → (يوضح القانون إذا كان في 3D)
المثال .

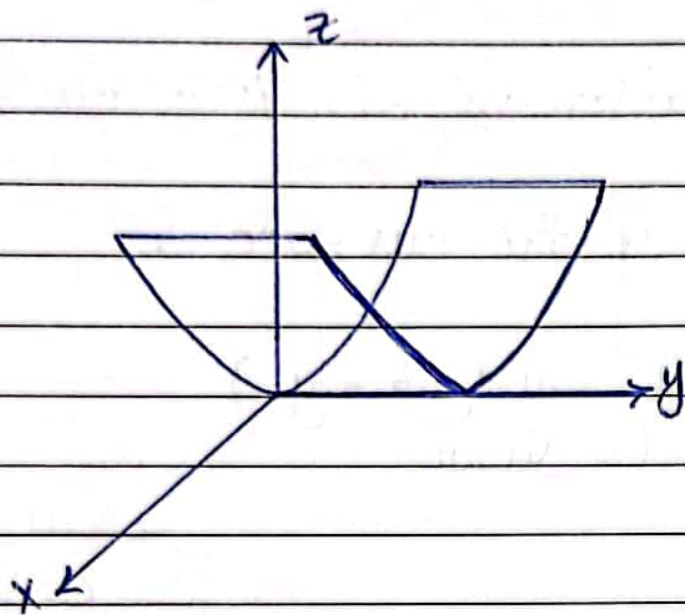


الرسم المفقود
طرح خط (صواب)
أي أنه المحاور المفقود
نحجب (صواب)

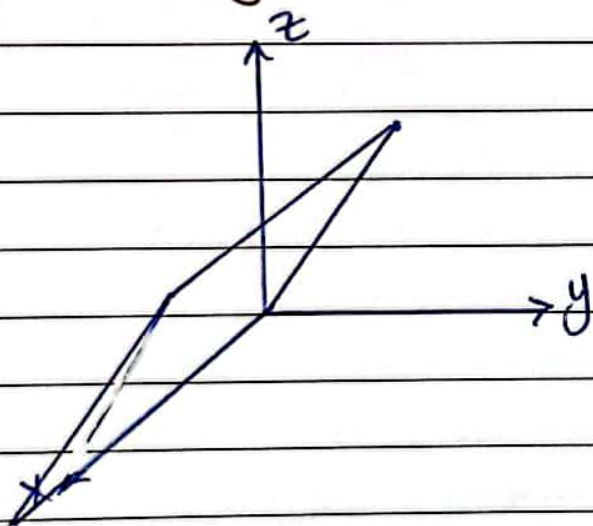
② $y = z^2$



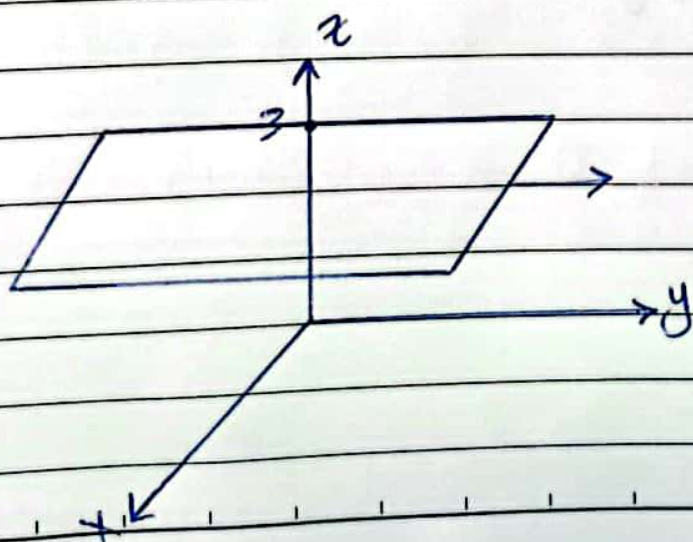
③ $z = x^2$



④ $z = y$



⑤ $z = 3 \rightarrow z = 3 + 0y$



plane fig

$\left\{ \begin{array}{l} x = \text{con} \\ y = \text{con} \\ z = \text{con} \end{array} \right\}$
 $\Downarrow$
 3D plane

The equation of xy -plane is $z=0$

The equation of xz -plane is $y=0$

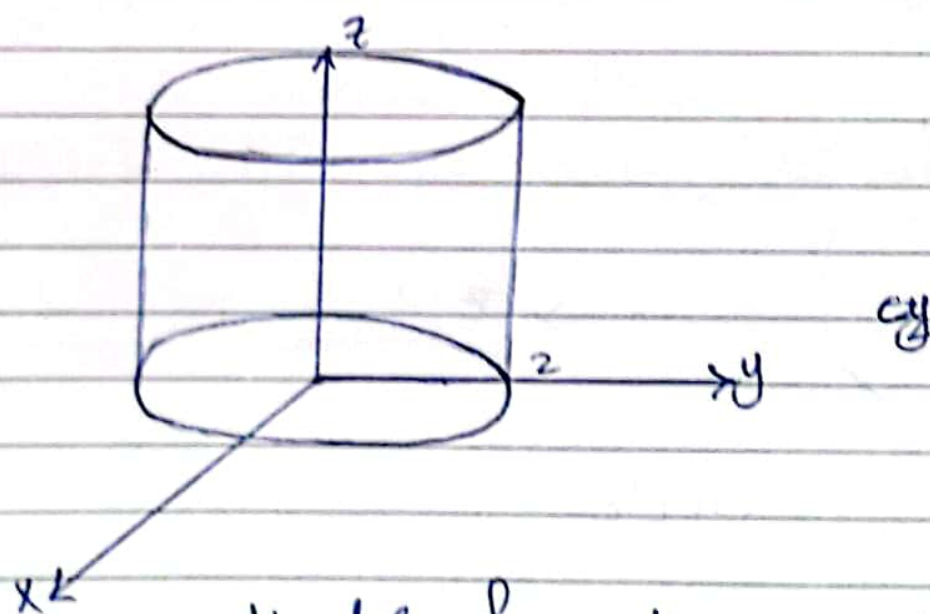
The equation of yz -plane is $x=0$

In the plane, the equation of the circle is

$$(x-a)^2 + (y-b)^2 = r^2$$

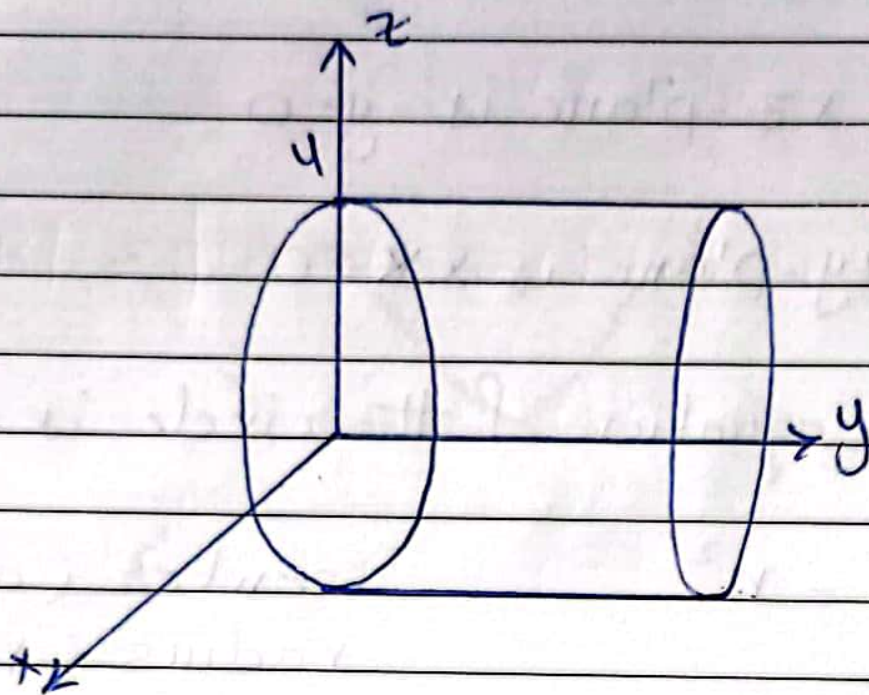
center (a,b)
radius (r)

Ex. ① Identify & sketch the graph of the equation $x^2 + y^2 = 4$ in $\mathbb{R}^3$.

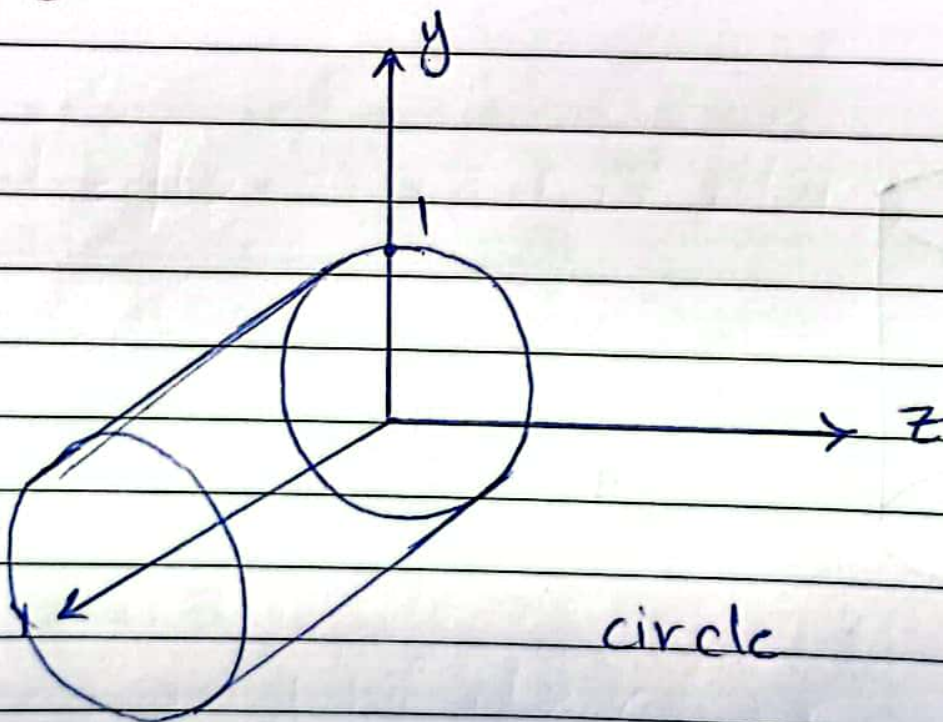


cylinder of radius 2 with the z -axis of symmetry

$$(2) \quad x^2 + z^2 = 16$$



$$(3) \quad y^2 + z^2 = 1$$

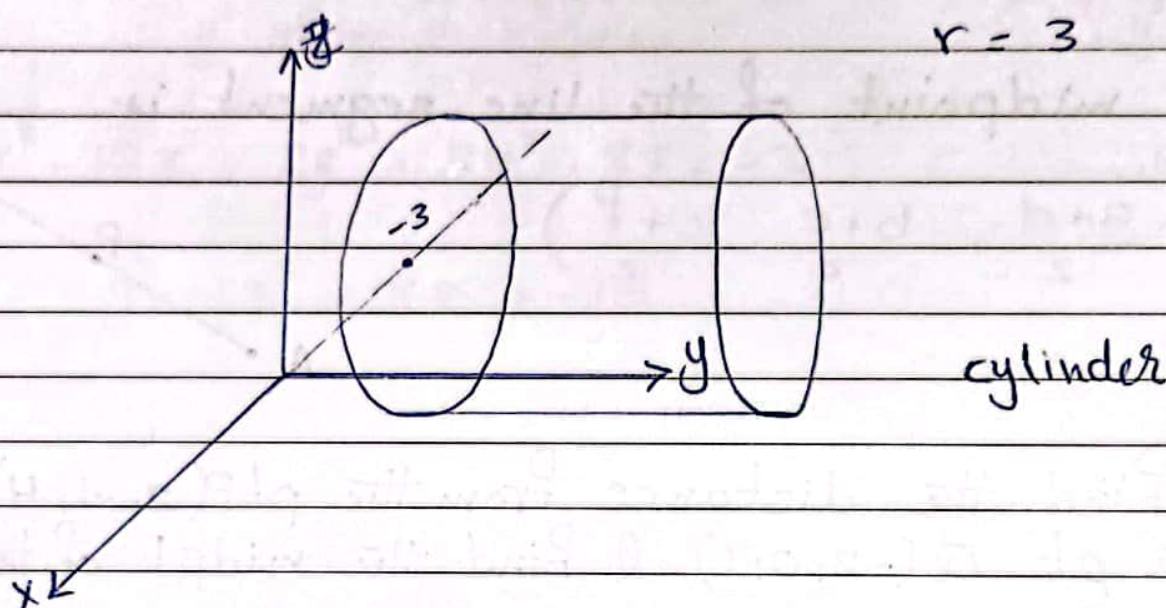


④ $x^2 + 6x + z^2 = 0 \rightarrow$ يجب كتابته بالشكل $x^2 + 6x + 9 + z^2 = 9$

$x^2 + 6x + 9 + z^2 = 9$

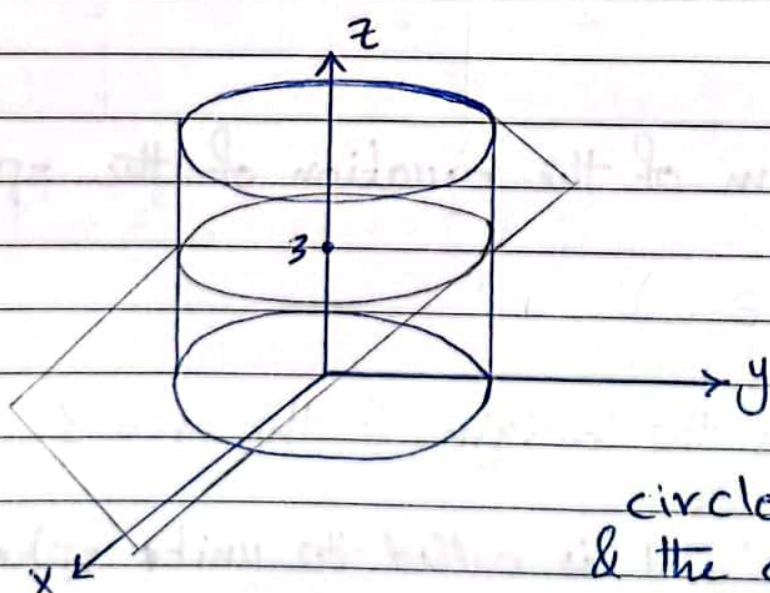
$(x+3)^2 + z^2 = 9$

the center is $(-3, 0)$
 $r = 3$



⑤ Which pt (x, y, z) satisfy the equation $x^2 + y^2 = 4$
 $z = 3$

(أ) نرسم surface الاول
 (ب) نرسم surface الثاني



circle of radius 2
 & the center is $(0, 0, 3)$

نصف الدائرة = نصف الخطين

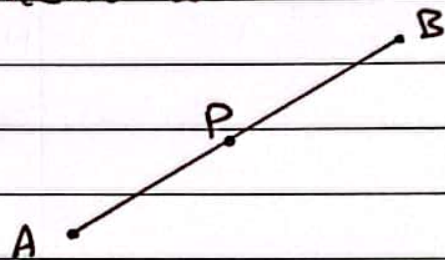
IF $A(a, b, c)$ & $B(d, e, f)$

1) the distance between A & B is

$$|AB| = \sqrt{(a-d)^2 + (b-e)^2 + (c-f)^2}$$

2) the midpoint of the line segment is

$$P\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right)$$



Ex: Find the distance from the pt $P(2, -1, 4)$ to the pt $Q(-2, 0, 1)$ & Find the midpt of the line

$$|PQ| = \sqrt{(2+2)^2 + (-1-0)^2 + (4-1)^2} = \sqrt{26}$$

$$\text{The midpt is } \left(\frac{2-2}{2}, \frac{-1+0}{2}, \frac{4+1}{2}\right) = \left(0, -\frac{1}{2}, \frac{5}{2}\right)$$

the standard form of the equation of the sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

when the center is the origin & the $r=1$ then

the sphere $x^2 + y^2 + z^2 = 1$ is called the unit sphere

$x^2 + y^2 = 1 \rightarrow$ not a sphere $\rightarrow$ z^2 لا يوجد

$$(4) (2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6) \div 2$$

$$x^2 - 6x + y^2 + z^2 + 4z = 3$$

$$(x-3)^2 + y^2 + (z+2)^2 = 16$$

The equation is a sphere its center $(3, 0, -2)$
& its radius 4

(5) Find the equation of the sphere centered at $A(0, -2, 5)$ of radius $\sqrt{3}$

$$\Rightarrow x^2 + (y+2)^2 + (z-5)^2 = 3$$

(6) Find the equation of the sphere if one of its diameters has end point $P(2, 1, 4)$ & $Q(2, -3, 0)$

$$\text{the distance is } \sqrt{(2-2)^2 + \left(\frac{-3-1}{2}\right)^2 + (4-0)^2} = \sqrt{32}$$

$$\Rightarrow \text{the radius is } \frac{\sqrt{32}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$$\Rightarrow \text{the center is } \left(\frac{2+2}{2}, \frac{1-3}{2}, \frac{4+0}{2}\right) = (2, -1, 2)$$

the equation is :

$$(x-2)^2 + (y+1)^2 + (z-2)^2 = 8$$

⑦ Find the equation of the sphere in the First octant of radius 5 that touch the coordinate planes

the center is $(5, 5, 5)$

∴ the equation is $(x-5)^2 + (y-5)^2 + (z-5)^2 = 25$

⑧ # The distance from the point $A(x, y, z)$ to the:

xy -plane is $|z|$ / xz -plane is $|y|$ / yz -plane is $|x|$

Ex: Find the equation of the largest sphere in the First octant centered at the pt $A(3, 2, 5)$

$$D_1 = \text{Dist}(A, xy\text{-plane}) = 5$$

$$D_2 = \text{Dist}(A, xz\text{-plane}) = 2$$

$$D_3 = \text{Dist}(A, yz\text{-plane}) = 3$$

~~distance is 3~~

$$(r = 2)$$

∴ the equation is $(x-3)^2 + (y-2)^2 + (z-5)^2 = 4$

Ex: Find the equation of the sphere centered at $A(1, -2, -5)$ & touches the xz -plane

$$r = \text{dis}(A, xz\text{-plane}) = |-2| = 2$$

في هذه الحالة r هو
المسافة بين
المركز ونقطة التماس

The equation is:

$$(x-1)^2 + (y+2)^2 + (z+5)^2 = 4$$

rule:

The distance from point $A(x, y, z)$ to the

x -axis is $\sqrt{y^2 + z^2}$

y -axis is $\sqrt{x^2 + z^2}$

z -axis is $\sqrt{y^2 + x^2}$

Ex: Find the distance from the pt $A(1, 4, -3)$ to the axis:

1) $(A, x\text{-axis}) = \sqrt{4^2 + (-3)^2} = 5$

2) $(A, y\text{-axis}) = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

3) $(A, z\text{-axis}) = \sqrt{1^2 + 4^2} = \sqrt{17}$

What the region in R^3 is represented by the equations:
inequalities:

1) $x^2 + y^2 + z^2 > 6z$

$$x^2 + y^2 + z^2 = 6z$$

$$x^2 + y^2 + z^2 - 6z + 9 = 9$$

$$x^2 + y^2 + (z - 3)^2 = 9$$

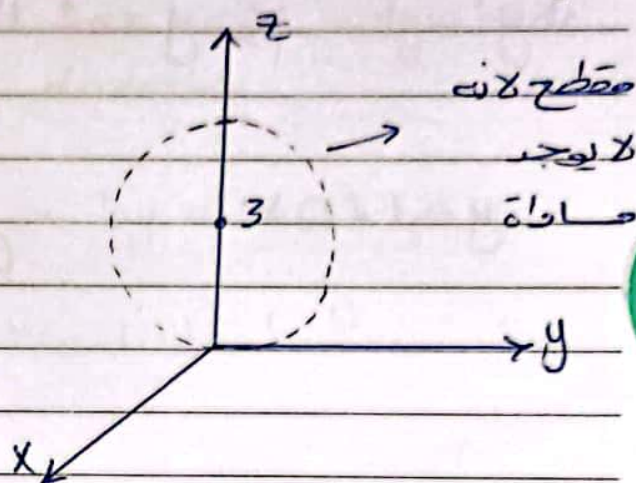
the center $(0, 0, 3)$

$$r = 3$$

لتحديد المنطقة يجب ان نأخذ نقطة اختبار (لا يجب ان تكون على المعادلة)

$(0, 0, 3)$ نأخذ $\rightarrow 0 + 0 + 9 > 18$ ✗

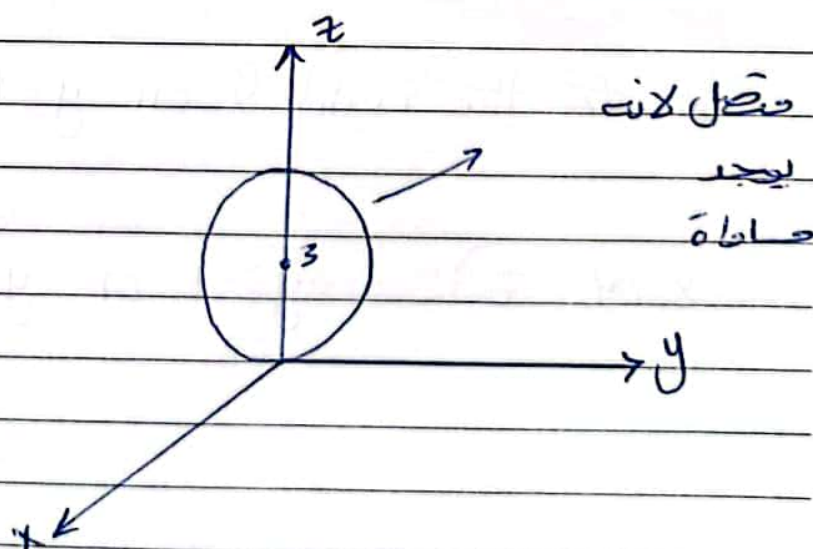
the region is outside the sphere



② $x^2 + y^2 + z^2 \leq 6z$

$$x^2 + y^2 + z^2 = 6z$$

$$x^2 + y^2 + (z - 3)^2 = 9$$



$(0, 0, 3)$ نأخذ $\rightarrow 0 + 0 + 9 \leq 18$ ✓

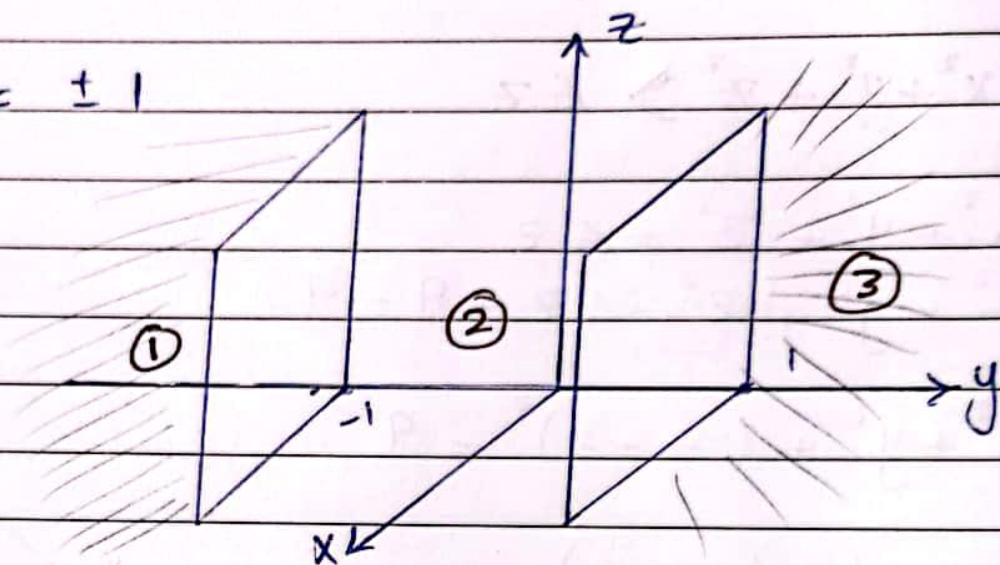
the region is inside the sphere
and on

اذا وجدت
دائرة
نضع (on)

$$\textcircled{3} \quad y^2 \geq 1$$

$$y^2 = 1 \rightarrow y = \pm 1$$

$$y = 1 + 0x$$



$$1) (0, -4, 0) \rightarrow 16 \geq 1 \quad \checkmark$$

$$2) (0, 0, 0) \rightarrow 0 \geq 1 \quad \times$$

$$3) (0, 4, 0) \rightarrow 16 \geq 1 \quad \checkmark$$

the region is to the left & on plane $y = -1$

& to the right & on $y = 1$

$$\{(x, y, z) : y \geq 1 \text{ or } y \leq -1\}$$

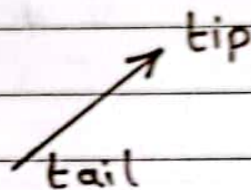
12.2) Vectors

A vector is a quantity that has both magnitude (length) written as $|\vec{v}|$ and direction.

A graph of a vector is given by a row ~~is~~

1) the magnitude of a vector is the distance from its tail to its tip

2) the direction is indicated by the row.



If we move from pt. A to pt. B then the displacement vector ~~is given by~~ is given by $\vec{v} = \overrightarrow{AB}$
 $|\vec{v}| = \text{dis}(A, B)$

When we write $\vec{v} = \overrightarrow{AB}$ then the point A is called the initial point & B the terminal point.

The zero vector $\vec{0}$ can be defined as a vector for which its initial & terminal are the same

السُّمِّيَ الصَّفْرِي = الصُّمِّي الصُّمِّي #

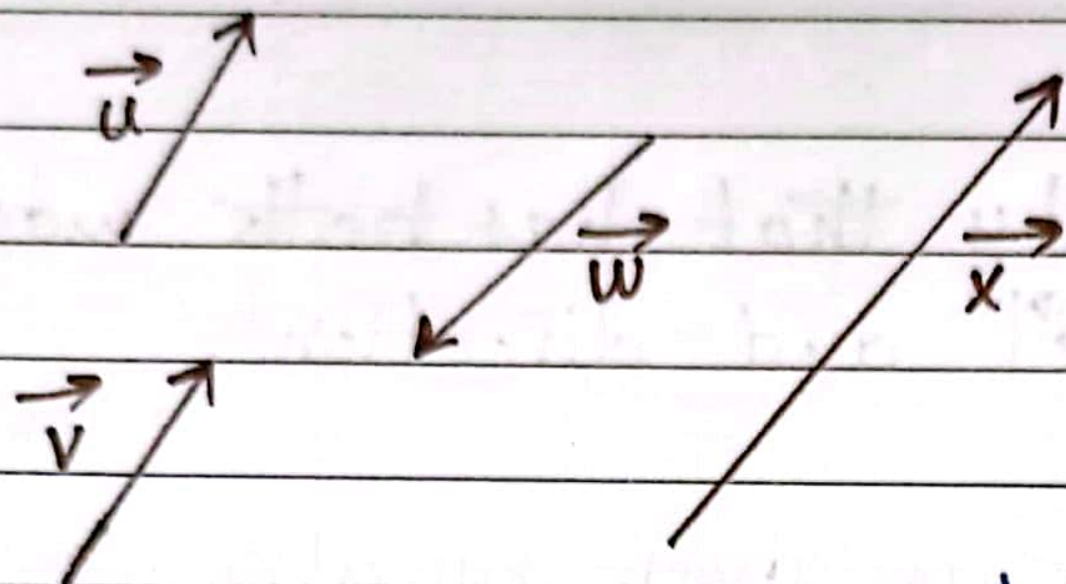
$$\overrightarrow{AA} = \overrightarrow{BB} = \overrightarrow{CC} = \vec{0}$$

$$|\overrightarrow{AA}| = |\overrightarrow{BB}| = |\overrightarrow{CC}| = 0$$

$$\vec{u} = \vec{v}$$

Two vectors $\vec{u}$ & $\vec{v}$ (are equal) when they have the same magnitude & direction

Ex :



- $\vec{u} = \vec{v}$ (the same ^{length} ~~magnitude~~ & direction)
- $\vec{u} \neq \vec{w}$ (different ^{pp} direction)
- $\vec{v} \neq \vec{x}$ (different ^{pp} length)

c is a scalar & $\vec{v}$ is a vector then $c\vec{v}$ is a vector of:

Length $|c\vec{v}| = |c| |\vec{v}|$
 ← $سعة المتجه$ → magnitude (vector)

the direction is : $\begin{cases} \text{in the same direction of } \vec{v} \text{ if } c > 0 \\ \text{in the opposite direction of } \vec{v} \text{ if } c < 0 \end{cases}$

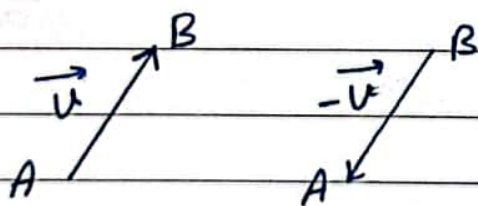
Ex: If $\vec{v}$ is the vector given with $|\vec{v}| = 3$

Find the lengths of these vectors: $2\vec{v}$, $-2\vec{v}$, $\frac{1}{2}\vec{v}$, $-\frac{1}{2}\vec{v}$

$|2\vec{v}| = 2|\vec{v}| = 6$ / $|-2\vec{v}| = 2|\vec{v}| = 6$ → هنا تكبر
 اتجاه المتجه

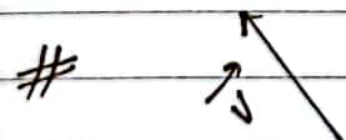
$|\frac{1}{2}\vec{v}| = \frac{1}{2}|\vec{v}| = \frac{3}{2}$ / $|\frac{-1}{2}\vec{v}| = \frac{1}{2}|\vec{v}| = \frac{3}{2}$ → هنا تقل
 الاتجاه

If $\vec{v} = \overrightarrow{AB}$ then $(-\vec{v} = \overrightarrow{BA})$ also $|\vec{v}| = |-\vec{v}|$
 $-\overrightarrow{AB} = \overrightarrow{BA}$

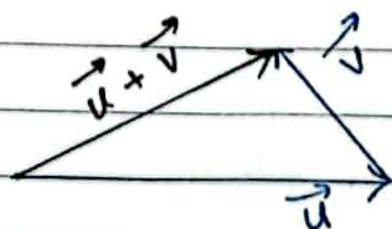


note:
 $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

نفس الاتجاه



Find $\vec{u} + \vec{v}$

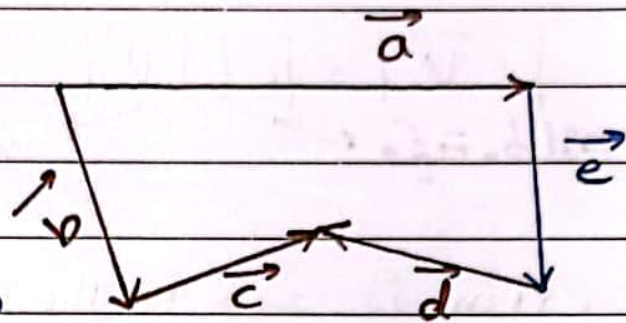


الرسم لكي نجعل او نطرح يجب ان يكون

انهاية ① = بداية ②

Ex: write the vector $\vec{e}$ as a sum of the vectors $\vec{a}, \vec{b}, \vec{c}$ in the figure

$$\vec{e} = -\vec{a} + \vec{b} + \vec{c} - \vec{d}$$



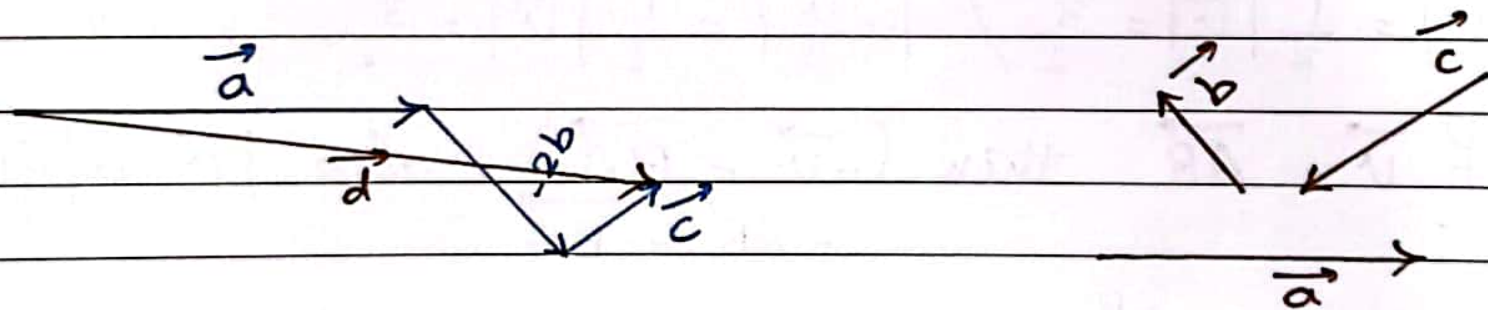
هذا ليس الحل الصحيح
وصولا الى الحل

Ex: Write $\vec{AB} + \vec{BC} - \vec{AC}$ in (implicit form)
simplicit

$$\vec{AB} + \vec{BC} + \vec{CA} = \vec{AC} + \vec{CA} = \vec{AA} = \vec{0}$$

الخط صفر

Ex: Draw the vector $\vec{a} - 2\vec{b} - \frac{1}{2}\vec{c}$



Component Form of vectors

Let $A(a, b, c)$ & $B(d, e, f)$ then: $\langle \quad \rangle$
هذا الرمز يعني متجه

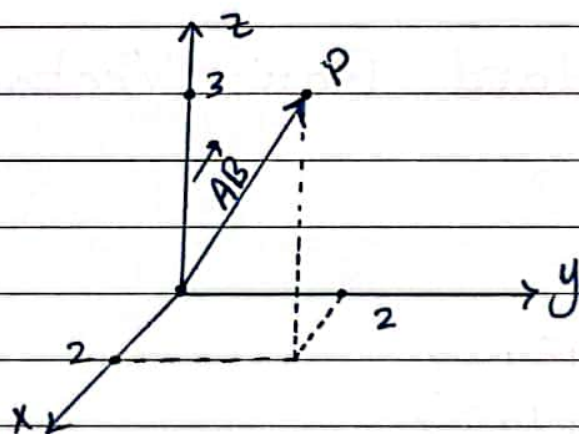
$$\vec{u} = \overrightarrow{AB} = \langle B - A \rangle = \langle d - a, e - b, f - c \rangle$$

$\langle d - a, e - b, f - c \rangle$ called the component of $\vec{u}$ $(\quad)$
امامنا يعني نقطة

Ex: Find & sketch the vector $\overrightarrow{AB}$ where
 $A(1, -1, -3)$ & $B(3, 1, 0)$

$$\overrightarrow{AB} = \langle B - A \rangle = \langle 3 - 1, 1 - (-1), 0 - (-3) \rangle = \langle 2, 2, 3 \rangle$$

$$p = (2, 2, 3)$$



$\vec{u} = \langle a, b, c \rangle$ then $|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$ in $\mathbb{R}^3$

$\vec{u} = \langle a, b \rangle$ then $|\vec{u}| = \sqrt{a^2 + b^2}$ in $\mathbb{R}^2$

Ex:

$$\vec{u} = \langle 2, -2, 1 \rangle \rightarrow |\vec{u}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

$$\vec{u} = \langle -5, \sqrt{11} \rangle \rightarrow |\vec{u}| = \sqrt{(-5)^2 + 11} = \sqrt{36} = 6$$

Ex: $\vec{a} = \langle -1, 0, 3 \rangle$ & $\vec{b} = \langle 2, -1, 5 \rangle$

Find $\left| 2\vec{a} - \frac{\vec{b}}{3} \right|$

First we find the component

$$2\vec{a} - \frac{\vec{b}}{3} = \left\langle 2(-1) - \frac{2}{3}, 2(0) + \frac{1}{3}, 2(3) - \frac{5}{3} \right\rangle$$

$$= \left\langle -\frac{8}{3}, \frac{1}{3}, -\frac{13}{3} \right\rangle$$

$$\left| 2\vec{a} - \frac{\vec{b}}{3} \right| = \sqrt{\frac{64}{9} + \frac{1}{9} + \frac{169}{9}} = \frac{\sqrt{234}}{3} = \sqrt{26}$$

Standard Basis Vectors

(In 3D)

$$\hat{i} = \langle 1, 0, 0 \rangle$$

$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle \rightarrow \langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

(In 2D)

$$\hat{i} = \langle 1, 0 \rangle$$

$$\hat{j} = \langle 0, 1 \rangle \rightarrow \langle a, b \rangle = a\hat{i} + b\hat{j}$$

Ex:

$$\textcircled{1} 5\hat{i} - \hat{j} - 7\hat{k} = \langle 5, -1, -7 \rangle$$

$$\textcircled{2} \frac{\hat{i}}{2} + 6\hat{k} = \left\langle \frac{1}{2}, 0, 6 \right\rangle$$

Ex: $\vec{a} = 5\hat{i} - \hat{j}$ & $\vec{b} = \langle 2, 4, -1 \rangle$ Find $|2\vec{a} + 3\vec{b}|$

First we find the vector

$$2\vec{a} + 3\vec{b} = 2\langle 5, -1, 0 \rangle + 3\langle 2, 4, -1 \rangle = \langle 16, 10, -3 \rangle$$

$$|2\vec{a} + 3\vec{b}| = \sqrt{256 + 100 + 9} = \sqrt{365}$$

A vector $\vec{v}$ is called a unit vector if $|\vec{v}| = 1$

Ex:

$$\textcircled{1} \vec{v} = \langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$$

$$|\vec{v}| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1 \quad \therefore \vec{v} \text{ is a unit vector}$$

$$\textcircled{2} \vec{v} = 0.5\hat{i} - 0.2\hat{j}$$

$$|\vec{v}| = \sqrt{0.25 + 0.04} = \sqrt{0.29} \neq 1 \quad \therefore \vec{v} \text{ is not a unit vector}$$

The zero vector is not a unit vector

Ex. Find the value of a that make $\vec{v} = \langle -\frac{1}{2}, \frac{1}{\sqrt{3}}, a \rangle$ a unit vector

$$|\vec{v}| = 1$$

$$1 = \sqrt{\frac{1}{4} + \frac{1}{3} + a^2} \rightarrow 1 = \frac{7}{12} + a^2 \rightarrow a^2 = \frac{5}{12}$$

$$\therefore a = \pm \frac{\sqrt{5}}{\sqrt{12}} = \pm \frac{\sqrt{5}}{2\sqrt{3}}$$

If $\vec{v} \neq \vec{0}$ then $\frac{\vec{v}}{|\vec{v}|}$ & $-\frac{\vec{v}}{|\vec{v}|}$ are two unit vectors

$\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the same direction

$-\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the opposite direction of $\vec{v}$

Ex: $\vec{v} = 2\hat{i} - 2\hat{j} + \hat{k}$

① Find the unit vector in the same direction of $\vec{v}$

$$\frac{\vec{v}}{|\vec{v}|} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

② Find a vector of length $\frac{3}{2}$ in the same direction

$$\frac{3}{2} \times \frac{\vec{v}}{|\vec{v}|} = \hat{i} - \hat{j} + \frac{3}{6}\hat{k}$$

③ Find a unit vector in the opposite direction of $\vec{v}$

$$-\frac{\vec{v}}{|\vec{v}|} = -\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

④ Find a vector of length $\sqrt{\pi}$ in the opposite direction

$$\sqrt{\pi} \left(-\frac{\vec{v}}{|\vec{v}|} \right) = -\frac{2\sqrt{\pi}}{3}\hat{i} + \frac{2\sqrt{\pi}}{3}\hat{j} - \frac{\sqrt{\pi}}{3}\hat{k}$$

Ex: Find all unit vector parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$

Let $\vec{v} = \langle 2, b \rangle$ be parallel to the tangent line stop

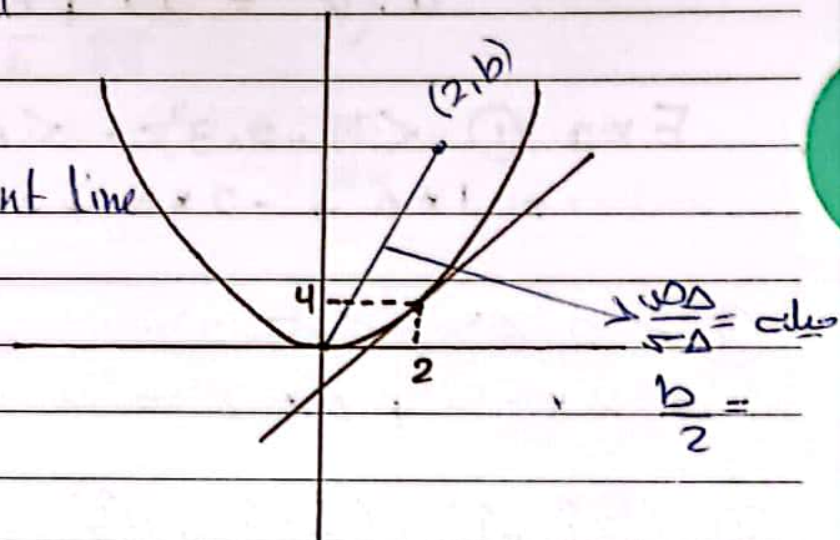
slope of $\vec{v}$ = slope of tangent line

$$\frac{b}{2} = 2 \times 1$$

$x=2$

$$\frac{b}{2} = 4 \rightarrow b = 8$$

$$\vec{v} = \langle 2, 8 \rangle$$



The unit vector

$$\frac{\vec{v}}{|\vec{v}|} = \frac{2\hat{i} + 8\hat{j}}{\sqrt{2^2 + 8^2}} = \frac{2\hat{i} + 8\hat{j}}{2\sqrt{17}} = \frac{1}{\sqrt{17}}\hat{i} + \frac{4}{\sqrt{17}}\hat{j}$$

$$-\frac{\vec{v}}{|\vec{v}|} = -\frac{1}{\sqrt{17}}\hat{i} - \frac{4}{\sqrt{17}}\hat{j}$$

Ex: 2

$$(1, 2) \in V_2 \quad \alpha \rightarrow (1, 2) \in \mathbb{R}^2.$$

$$\langle 1, 2 \rangle \in \mathbb{R}^2 \quad \alpha \rightarrow \langle 1, 2 \rangle \in V_2$$

12.3) The Dot Product

$$\vec{u} = \langle a, b, c \rangle \quad \& \quad \vec{v} = \langle d, e, f \rangle$$

then the dot product of $\vec{u}$ & $\vec{v}$

$$\vec{u} \cdot \vec{v} = ad + be + cf$$

Ex: ① $\langle 1, -2, 3 \rangle \cdot \langle 6, 3, 0 \rangle$

$$= 1 \times 6 + -2 \times 3 + 3 \times 0 = 0$$

② $\langle 2, 6 \rangle \cdot \langle -5, 2 \rangle$

$$= 2 \times -5 + 6 \times 2 = 2$$

③ $(3\hat{i} - \hat{j}) \cdot (-2\hat{i} + 4\hat{k})$

$$= 3 \times -2 + -1 \times 0 + 0 \times 4 = -6$$

The angle θ between two vectors $\vec{u}$ & $\vec{v}$ is the angle between them when the vectors have the same initial point

$\Rightarrow$ rule : $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$

IF $\vec{u} \neq 0$ & $\vec{v} \neq 0$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

Ex: Find the angle between the two vectors

$$\vec{u} = -\hat{i} + \hat{k} \quad \& \quad \vec{v} = 3\hat{i} + \hat{j} + \hat{k}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \left(\frac{-1 \times 3 + 0 \times 1 + 1 \times 1}{\sqrt{2} \sqrt{11}} \right)$$

$$\theta = \cos^{-1} \left(\frac{-2}{\sqrt{22}} \right) = 115.2^\circ \quad \text{in degree} \quad \neq \quad \frac{\pi}{180} = 2.01 \quad \text{in radian}$$

Ex: Find the value of x that makes the angle between the two vectors $\vec{u} = \langle 2, 1, -1 \rangle$ & $\vec{v} = \langle 1, x, 0 \rangle$

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \Rightarrow 2(1) + (x) + (-1)(0) \quad \text{is } \frac{\pi}{4}$$

$$= \sqrt{6} \sqrt{1+x^2} \cos \frac{\pi}{4} \Rightarrow 2+x = \frac{\sqrt{6} \sqrt{1+x^2}}{\sqrt{2}}$$

$$\Rightarrow (2+x)^2 = 3(1+x^2) \Rightarrow x^2 + 4x + 4 = 3 + 3x^2$$

$$2x^2 - 4x - 1 = 0 \Rightarrow x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 2 \times (-1)}}{2 \times 2} = \frac{4 \pm \sqrt{24}}{4}$$

$$= 1 \pm \frac{\sqrt{6}}{2}$$

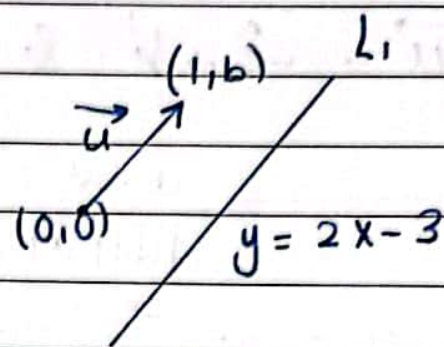
Ex: Find the angle between two lines in $\mathbb{R}^2$
 $2x - y = 3$ & $3x + y = 7$ خط L_1 = $\vec{u}$ و خط L_2 = $\vec{v}$

$\vec{u} \langle 1, b \rangle$ parallel to $L_1: 2x - y = 3$

slope of $\vec{u}$ = slope of L_1

$$b = 2$$

$$\vec{u} = \langle 1, 2 \rangle$$

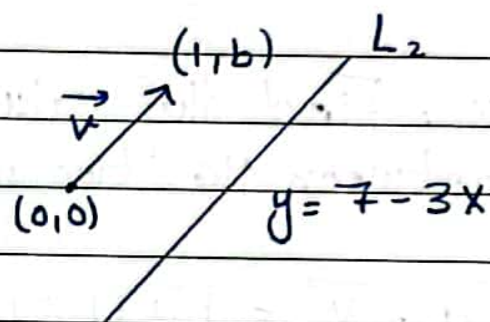


$\vec{v} \langle 1, b \rangle$ parallel to L_2

slope of $\vec{v}$ = slope of L_2

$$b = -3$$

$$\vec{v} = \langle 1, -3 \rangle$$



$$\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \cos^{-1} \left(\frac{-5}{\sqrt{5} \sqrt{10}} \right) \rightarrow \text{الزاوية منفرجة} = 135^\circ$$

$$\therefore \theta = 180 - 135 = 45^\circ$$

Ex: If $\vec{u}$ & $\vec{v}$ are vectors such that $|\vec{u}| = 4$

$|\vec{v}| = 3$ and the angle between $\vec{u}$ & $\vec{v} = \frac{2\pi}{3}$

① Find $\vec{u} \cdot \vec{v}$

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta = 4 \times 3 \cos \frac{2\pi}{3} = 12 \left(-\cos \frac{\pi}{3} \right) = -6$$

② Find $|2\vec{u} - 3\vec{v}|$

$$|\vec{a} + \vec{b}| = |\vec{a} + \vec{b}|^2$$

$$\begin{aligned} |2\vec{u} - 3\vec{v}|^2 &= 2^2 |\vec{u}|^2 - 2 \times 2 \times 3 \vec{u} \cdot \vec{v} + 3^2 |\vec{v}|^2 \\ &= 4 \times 16 - 12(-6) + 9 \times 9 \\ &= 217 \end{aligned}$$

~~Find~~

③ $|3\vec{u} + \vec{v}| = 117$

Ex: If $\vec{a}$ & $\vec{b}$ are vectors such that $|\vec{a}| = \sqrt{3}$
 $|2\vec{a} - 3\vec{b}| = \sqrt{45}$ & $|\vec{a} + 2\vec{b}| = \sqrt{27}$

المطلوب عاظمي magnitude متكوعه طريقه التربع

① Find $\vec{a} \cdot \vec{b}$

$$|2\vec{a} - 3\vec{b}|^2 = (\sqrt{45})^2 \Rightarrow 4|\vec{a}|^2 + 2 \times 2 \times 3 \vec{a} \cdot \vec{b} + 9|\vec{b}|^2 = 45$$

$$4 \times 3 - 12 \vec{a} \cdot \vec{b} + 9|\vec{b}|^2 = 45 \Rightarrow (-12 \vec{a} \cdot \vec{b} + 9|\vec{b}|^2 = 33) \div 3$$

$$-4 \vec{a} \cdot \vec{b} + 3|\vec{b}|^2 = 11 \rightarrow \textcircled{1}$$

$$|\vec{a} + 2\vec{b}|^2 = (\sqrt{27})^2 \Rightarrow |\vec{a}|^2 + 2 \times 2 \times \vec{a} \cdot \vec{b} + 4|\vec{b}|^2 = 27$$

$$4 \vec{a} \cdot \vec{b} + 4|\vec{b}|^2 = 24 \rightarrow \textcircled{2}$$

$$\textcircled{1} + \textcircled{2} \Rightarrow 7|\vec{b}|^2 = 35 \Rightarrow |\vec{b}|^2 = 5 \Rightarrow |\vec{b}| = \sqrt{5}$$

$$\textcircled{2} \quad 4 \vec{a} \cdot \vec{b} + 4(\sqrt{5})^2 = 24 \Rightarrow \vec{a} \cdot \vec{b} = 1$$

② Find the angle between $\vec{a}$ & $\vec{b}$

$$\theta = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \cos^{-1} \frac{1}{\sqrt{3} \sqrt{5}} = \cos^{-1} \frac{1}{\sqrt{15}} = 75.04^\circ = 1.31$$

③ Find $|\vec{a} + 2\vec{b}|$

$$|\vec{a} + 2\vec{b}|^2 = |\vec{a}|^2 + 2 \times 2 \vec{a} \cdot \vec{b} + 4|\vec{b}|^2 = 3 + 4 \times 1 + 4 \times 5 = 27$$

Ex: Prove that $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

$$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 + |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$= 2(|\vec{a}|^2 + |\vec{b}|^2)$$

Two vectors $\vec{u}$ & $\vec{v}$ are perpendicular or orthogonal written $\vec{u} \perp \vec{v} \Leftrightarrow \vec{u} \cdot \vec{v} = 0 \rightarrow \theta = 90^\circ$

معرفة إذا المتجهين متعامدين في dot product إذا كان الجواب $= 0$
إذا الزاوية 90° وبالتالي يكونان متعامدين إذا كان $\neq 0$ إذا غير متعامدين

Ex: show that $2\hat{i} + 2\hat{j} - \hat{k}$ is perpendicular to $5\hat{i} - 4\hat{j} + 2\hat{k}$

$$(2\hat{i} + 2\hat{j} - \hat{k}) \cdot (5\hat{i} - 4\hat{j} + 2\hat{k}) = 2 \cdot 5 + 2(-4) + (-1)2 = 0$$

$$\Rightarrow (2\hat{i} + 2\hat{j} - \hat{k}) \perp (5\hat{i} - 4\hat{j} + 2\hat{k})$$

Ex: Find the value of a that makes $a\hat{i} - 2\hat{j} + \hat{k}$ perpendicular to $2\hat{i} + a\hat{k} + \hat{j}$

$$(a\hat{i} - 2\hat{j} + \hat{k}) \cdot (2\hat{i} + \hat{j} + a\hat{k}) = 0$$

$$2a - 2 + a = 0$$

$$3a = 2 \rightarrow a = \frac{2}{3}$$

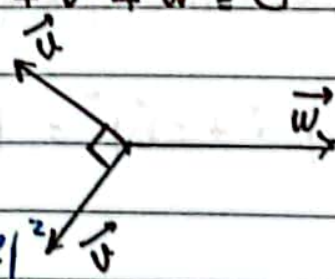
Ex: If $\vec{u}$ & $\vec{v}$ are unit vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$ then find $|\vec{w}|$ $|\vec{u}| = 1$, $|\vec{v}| = 1$

$$|\vec{u}| \perp |\vec{v}| \Leftrightarrow \vec{u} \cdot \vec{v} = 0$$

$$\vec{w} = -(\vec{u} + \vec{v})$$

$$|\vec{w}|^2 = |-(\vec{u} + \vec{v})|^2 = |\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + \vec{u} \cdot \vec{v} + |\vec{v}|^2$$

$$= 1 + 0 + 1 = 2$$



$$\# \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \Rightarrow \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

- 1) $\vec{u} \cdot \vec{v} > 0 \Rightarrow \theta$ is an acute angle
- 2) $\vec{u} \cdot \vec{v} < 0 \Rightarrow \theta$ is an obtuse angle
- 3) $\vec{u} \cdot \vec{v} = 0 \Rightarrow \theta$ is a right angle

Ex: The angle between the vectors $2\hat{i} - \hat{k}$ & $\hat{j} + 2\hat{k}$ is an obtuse angle

$$(2\hat{i} - \hat{k}) \cdot (\hat{j} + 2\hat{k}) = -2 < 0$$

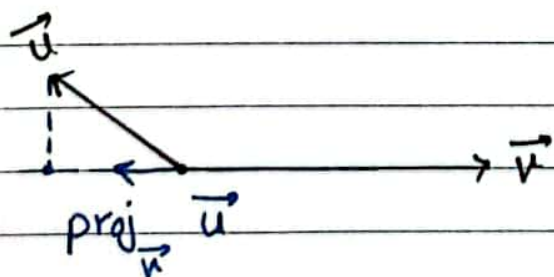
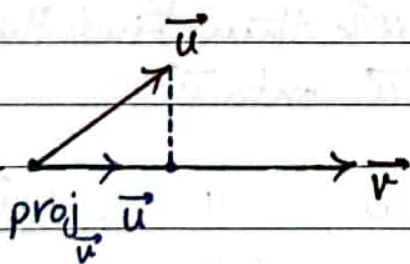
Types of projections

1) The scalar projection of the vector $\vec{u}$ onto the vector $\vec{v}$ is

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

2) The vector projection of the vector $\vec{u}$ onto the vector $\vec{v}$ is

$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$



$$\theta = 0$$

$\vec{u}, \vec{v}$ has the same direction

$$\text{proj } \vec{u} = \vec{u}$$

$$\theta = 180$$

$$\text{proj } \vec{u} = -\vec{u}$$

$$\theta = 90$$

$$\text{proj } \vec{u} = 0$$

$$\# \text{ comp}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta$$

$$\# \text{ proj}_{\vec{v}} \vec{u} = \left| \text{comp}_{\vec{v}} \vec{u} \right| \frac{\vec{v}}{|\vec{v}|}$$

Ex: Find the scalar & vector projection of $\vec{v} = \langle 1, 1, 2 \rangle$ onto $\vec{u} = -2\hat{i} + \hat{j} - \hat{k}$

$$\text{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} = \frac{1(-2) + 1(1) + 2(-1)}{\sqrt{4+1+1}} = \frac{-3}{\sqrt{6}} \quad \text{الزائفة مقترجة}$$

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u} = \frac{-3}{6} \vec{u} = -\frac{1}{2} (-2\hat{i} + \hat{j} - \hat{k}) = \hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

Ex: ① If $|\vec{u}| = 5$ & the angle between $\vec{u}$ & $\vec{v}$ is $\frac{5\pi}{6}$ then Find the scalar projection of $\vec{u}$ onto $\vec{v}$

$$\text{comp}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta = 5 \cos \frac{5\pi}{6} = 5 \left(-\cos \frac{\pi}{6} \right) = -\frac{5\sqrt{3}}{2}$$

② If $\text{comp}_{\vec{v}} \vec{u} = -4$ & $\vec{v} = 3\hat{j} - 4\hat{k}$ then Find the ^{vector} scalar projection of $\vec{u}$ onto $\vec{v}$

$$\text{proj}_{\vec{v}} \vec{u} = \left(\text{comp}_{\vec{v}} \vec{u} \right) \frac{\vec{v}}{|\vec{v}|} = -4 \frac{3\hat{j} - 4\hat{k}}{\sqrt{9+16}} = \frac{-12\hat{j} + 16\hat{k}}{5}$$

Ex: Show that $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$

$$\vec{v} \cdot (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) =$$

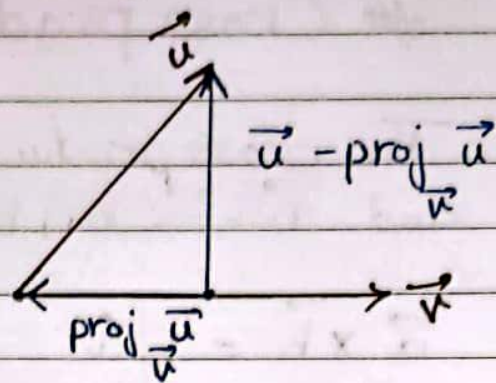
$$\vec{v} \cdot \vec{u} - \vec{v} \cdot \text{proj}_{\vec{v}} \vec{u} =$$

$$\vec{u} \cdot \vec{v} - \vec{v} \cdot \left(\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \right) =$$

$$\vec{u} \cdot \vec{v} - \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v} \cdot \vec{v} =$$

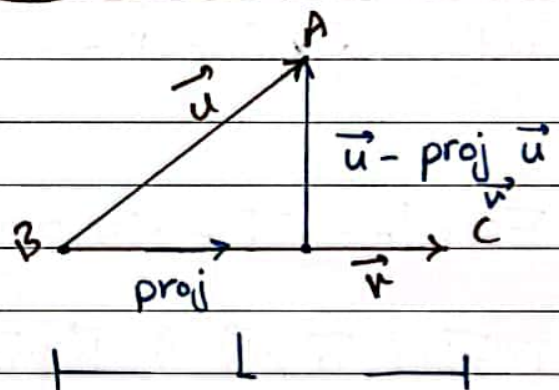
$$\vec{u} \cdot \vec{v} - \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) |\vec{v}|^2 = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0$$

$$\therefore (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) \perp \vec{v}$$



the distance from the point A
to the line L is

$$|\vec{u} - \text{proj}_{\vec{v}} \vec{u}|_{\vec{u}} = \overline{BA} \text{ \& } \vec{v} = \overline{BC}$$



Ex: Find the distance from the point A (1, 2, 3) & the line
that pass through the points B (2, 1, 3) & C (0, 1, 0)

$$\vec{u} = \overline{BA} = \langle A - B \rangle = \langle -1, 1, 0 \rangle$$

$$\vec{v} = \overline{BC} = \langle C - B \rangle = \langle -2, 0, -3 \rangle$$

$$\vec{u} - \text{proj}_{\vec{v}} \vec{u} = \vec{u} - \left(\frac{2}{13} \right) \vec{v} = \left\langle -1 - \frac{2}{13}(-2), 1 - \frac{2}{13}(0), 0 - \frac{2}{13}(-3) \right\rangle$$

$$= \left\langle -\frac{9}{13}, 1, \frac{6}{13} \right\rangle$$

$$\Rightarrow \text{the distance} = |\vec{u} - \text{proj}_{\vec{v}} \vec{u}| = \frac{\sqrt{286}}{13}$$

12.4)

Cross product

The cross product of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is given by:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \hat{i} - (a_1 b_3 - a_3 b_1) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}$$

To find $\vec{a} \times \vec{b}$ we must have a & b in V_3

To find $\vec{a} \cdot \vec{b}$ we must have a & b in V_2 or V_3

$\vec{a} \times \vec{b}$ is a vector orthogonal to both $\vec{a}$ & $\vec{b}$ and so $\vec{a} \times \vec{b}$ orthogonal to the plane containing both $\vec{a}$ & $\vec{b}$

Ex: let $\vec{a} = \langle 3, 2, 1 \rangle$ & $\vec{b} = \langle -1, 1, 0 \rangle$

① Find $\vec{a} \times \vec{b}$ & $\vec{b} \times \vec{a}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -\hat{i} - \hat{j} + 5\hat{k}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \hat{i} + \hat{j} - 5\hat{k}$$

② Find two vectors perpendicular to both $\vec{a}$ & $\vec{b}$

$$\vec{a} \times \vec{b} \text{ \& } -\vec{a} \times \vec{b} \Rightarrow -\hat{i} - \hat{j} + 5\hat{k} \text{ \& } \hat{i} + \hat{j} - 5\hat{k}$$

③ Find two unit vectors orthogonal to both $\vec{a}$ & $\vec{b}$

$$\frac{-\hat{i} - \hat{j} + 5\hat{k}}{|-\hat{i} - \hat{j} + 5\hat{k}|} = \frac{-\hat{i} - \hat{j} + 5\hat{k}}{\sqrt{27}}$$

$$\frac{\hat{i} + \hat{j} - 5\hat{k}}{|\hat{i} + \hat{j} - 5\hat{k}|} = \frac{\hat{i} + \hat{j} - 5\hat{k}}{\sqrt{27}}$$

④ Find two unit vectors orthogonal to the plane that pass through the points $A(1,2,3)$ $B(4,4,4)$ $C(0,3,3)$

$$\vec{a} = \vec{AB} = \langle B - A \rangle = \langle 3, 2, 1 \rangle$$

$$\vec{b} = \vec{AC} = \langle C - A \rangle = \langle -1, 1, 0 \rangle$$

$\vec{a} \times \vec{b}$ & $\vec{b} \times \vec{a}$ are orthogonal to both $\vec{a}$ & $\vec{b}$

$\frac{-\hat{i} - \hat{j} + 5\hat{k}}{\sqrt{27}}$ & $\frac{\hat{i} + \hat{j} - 5\hat{k}}{\sqrt{27}}$ are unit vectors orthogonal to the plane $\vec{a}$ & $\vec{b}$

(note)

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{i} \times \hat{k} = -\hat{j}$$

properties of Cross product

- 1) $\vec{u} \times \vec{u} = 0$
- 2) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- 3) $\vec{0} \times \vec{v} = \vec{u} \times \vec{0} = 0$
- 4) $(a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v}) = a(\vec{u} \times \vec{v})$
- 5) $\vec{u} (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
- 6) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

$$\# \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

let $\vec{a}$ & $\vec{b}$ orthogonal

Ex: $|\vec{a}| = 2$ & $|\vec{b}| = 3$ find $(\vec{b} \times \vec{a}) \times \vec{a}$ & $|(\vec{b} \times \vec{a}) \times \vec{a}|$

$$(\vec{b} \times \vec{a}) \times \vec{a} = -\vec{a} \times (\vec{b} \times \vec{a}) = -((\vec{a} \cdot \vec{a}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{a})$$

$$= -(|\vec{a}|^2 \vec{b} - 0 \vec{a}) = -4\vec{b}$$

$$|(\vec{b} \times \vec{a}) \times \vec{a}| = |-4\vec{b}| = +4|\vec{b}| = 12$$

Ex: Simplify $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$$\begin{aligned} (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) &= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b} \\ &= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0} \\ &= 2\vec{a} \times \vec{b} \end{aligned}$$

the length of $\vec{a} \times \vec{b}$ is given by $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$

$$|\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}$$

↓
 $\sin \theta$
 since

Ex: if $\vec{a} = \langle 2, -1, 0 \rangle$ & $\vec{b} = \langle 3, 0, 4 \rangle$ then $|\vec{a} \times \vec{b}|$

$$|\vec{a}| = \sqrt{25} = 5$$

$$|\vec{b}| = \sqrt{25} = 5$$

$$|\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2} = \sqrt{5 \times 25 - (36)} = \sqrt{89}$$

the distance from point A to the line $l = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|}$
where $\vec{u} = \vec{BA}$ & $\vec{v} = \vec{BC}$ → line passing through B

Ex: Find the distance from the point A(1, 2, 3) & the line that pass through the points B(2, 1, 3) & C(0, 1, 0)

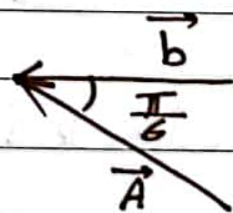
$$|\vec{u}| = \vec{BA} = \langle A - B \rangle = \langle -1, 1, 0 \rangle$$

$$|\vec{v}| = \vec{BC} = \langle A - B \rangle = \langle -2, 0, -3 \rangle$$

$$\text{Distance} = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|} = \frac{\sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}}{|\vec{u}|} = \frac{\sqrt{22}}{\sqrt{13}}$$

Ex. Find $|\vec{a} \times \vec{b}|$ where $|\vec{a}| = 8$, $|\vec{b}| = 6$

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta = 8 \times 6 \times \sin \frac{\pi}{6} = 24$$



Ex. Find $|\vec{a} \times \vec{b}|$ & $\vec{a} \times \vec{b}$ where $|\vec{a}| = 2$ & $|\vec{b}| = \frac{1}{2}$
and $|\vec{a} + 2\vec{b}| = 3$

$$|\vec{a} + 2\vec{b}|^2 = 3^2 \Rightarrow |\vec{a}|^2 + 2 \times 2 \times \vec{a} \cdot \vec{b} + 4|\vec{b}|^2 = 9$$
$$\Rightarrow \vec{a} \cdot \vec{b} = 1$$

$$|\vec{a} \times \vec{b}| = \sqrt{4\left(\frac{1}{4}\right) - 1^2} = 0 \Rightarrow \vec{a} \times \vec{b} = 0$$

two vectors $\vec{a}$ & $\vec{b}$ are parallel if $\vec{a} \times \vec{b} = \vec{0}$

If $\vec{a}$ any vector then $\vec{a} \parallel \vec{0}$ since $\vec{a} \times \vec{0} = \vec{0}$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \text{ \& } \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\text{then } \vec{a} \parallel \vec{b} \text{ if } \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$\text{Ex: } \textcircled{1} \langle 6, 3, 15 \rangle \parallel \langle 4, 2, 10 \rangle$$

$$\text{since } \frac{6}{4} = \frac{3}{2} = \frac{15}{10} \text{ are all equal}$$

$$\textcircled{2} \langle 4, 6, -28 \rangle \text{ \& } \langle 2, 3, 14 \rangle \text{ are not parallel}$$

$$\text{since } \frac{4}{2}, \frac{6}{3}, \frac{-28}{14} \text{ are not equal}$$

Ex: Find the value of x that makes $\vec{a} = \langle 2, x-1, x \rangle$
and $\vec{b} = \langle x^2-1, 0, x+1 \rangle$ parallel

$$\frac{x^2-1}{2} = \frac{0}{x-1} = \frac{x+1}{x} \Rightarrow 0 = \frac{x+1}{x} \rightarrow x+1=0 \Rightarrow x=-1$$

$$\text{Check: } \frac{0}{2} = \frac{0}{-2} = \frac{0}{-1} \text{ no error}$$

$\therefore$ the value of $x = -1$

Ex: Find the value of x that makes $\vec{a} = \langle 3, 1, x^2+2x+1 \rangle$
& $\vec{b} = \langle 3x^2-3, 3, 3 \rangle$ parallel

$$x = -2$$

~~There~~ Three points A, B, C are collinear (نقطة على استقامة)
if $\overrightarrow{AB} \parallel \overrightarrow{AC}$

Ex. Determine whether the points A(2, 4, -3)
B(3, -1, 1), C(4, -6, 5) are collinear or not.

$$\overrightarrow{AB} = \langle 1, -5, 4 \rangle \quad \overrightarrow{AC} = \langle 2, -10, 8 \rangle$$

$$\frac{2}{1} = \frac{-10}{-5} = \frac{8}{4} \text{ are } \dots \text{ all equal}$$

$$\therefore \overrightarrow{AB} \parallel \overrightarrow{AC} \Rightarrow \text{The points are collinear.}$$

The area of the parallelogram determined by the vectors $\vec{a}$ & $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$

The area of the triangle determined by the vectors $\vec{a}$ & $\vec{b}$ is $A = \frac{1}{2} |\vec{a} \times \vec{b}|$

Let A, B, C, D points & let $\vec{a} = \overrightarrow{AB}$ & $\vec{b} = \overrightarrow{AC}$

$$\text{The area of parallelogram} = |\vec{a} \times \vec{b}|$$

$$\text{The area of triangle} = \frac{1}{2} |\vec{a} \times \vec{b}|$$

Ex: let $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ & $\vec{b} = \hat{j} + 3\hat{k}$

$A(1,0,1)$ $B(2,2,0)$ $C(1,1,4)$ $D($

① Find the area of the parallelogram

$$A = |\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2} = \sqrt{59}$$

② Find the area of the triangle

$$A = \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{\sqrt{59}}{2}$$

③ Find the area of parallelogram with A, B, C, D

$$\vec{a} = \overrightarrow{AB} = \langle 1, 2, -1 \rangle \text{ & } \vec{b} = \overrightarrow{AC} = \langle 0, 1, 3 \rangle$$

$$A = |\vec{a} \times \vec{b}| = \sqrt{59}$$

④

$$A = \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{\sqrt{59}}{2}$$

$$\# \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - c_3b_1) + a_3(b_1c_2 - b_2c_1)$$

$$\# \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

Ex. if $\vec{a} \cdot (\vec{b} \times \vec{c}) = -3$ find $2\vec{c} \cdot (\vec{b} \times 3\vec{a})$

$$2\vec{c} \cdot (\vec{b} \times 3\vec{a}) = 2 \times 3 \vec{c} \cdot (\vec{b} \times \vec{a}) = 6 (-\vec{a} \cdot (\vec{b} \times \vec{c})) = -6 \times -3 = 18$$

The volume of the parallelepiped determined by the vectors $\vec{a}, \vec{b}, \vec{c}$ is

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

$\swarrow \quad \searrow$
a b c

Ex. Find the volume of the parallelepiped

① $\vec{a} = \langle 0, -2, 5 \rangle$, $\vec{b} = \langle 0, 1, 2 \rangle$, $\vec{c} = \langle 6, 3, -1 \rangle$

$$\textcircled{1} \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 0 & -2 & 5 \\ 0 & 1 & 2 \\ 6 & 3 & -1 \end{vmatrix} = -54$$

$$V = |-54| = 54$$

② $P(-2, 1, 0)$, $Q(2, -1, 5)$, $R(-2, 2, 2)$, $S(4, 4, -1)$
 PQ, PR, PS

$$PQ = \langle 0, -2, 5 \rangle \quad PR = \langle 0, 1, 2 \rangle \quad PS = \langle 6, 3, -1 \rangle$$

the volume = 54

Three vectors $\vec{a}, \vec{b}, \vec{c}$ in V_3 are coplanar (lie in the same plane) if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

4 point A, B, C, D are coplanar if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$
where $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AC}$, $\vec{c} = \overrightarrow{AD}$

Ex: Find the value of x that makes $\vec{a} = \langle 1, x, 0 \rangle$
 $\vec{b} = \langle x, 2, 1 \rangle$, $\vec{c} = \langle 0, 1, 1 \rangle$ coplanar

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & x & 0 \\ x & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \pm 1 \Rightarrow x = \pm 1$$

② Find the value of x that makes the points $A(1, -1, 2)$
 $B(2, x-1, 2)$ $C(x+1, 1, 3)$ $D(1, 0, 3)$ lie in the
same plane

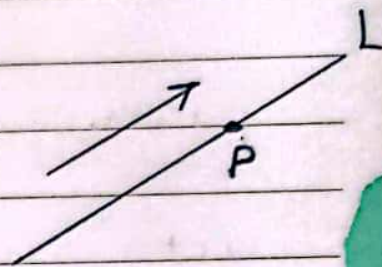
$$x = \pm 1$$

12.5 (Equations of lines and planes)

Let L be the line that pass through the pt $P(x_0, y_0, z_0)$ and parallel to the vector $\langle a, b, c \rangle$

① The parametric equations of L are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$



نائب نقطة يمر
line
direction vectors

② The symmetric equations of L are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

if $a \neq 0 \rightarrow$ The symm. eqs are $\frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad x = x_0$

if $b \neq 0 \rightarrow$ The symm. eqs are $\frac{x - x_0}{a} = \frac{z - z_0}{c}, \quad y = y_0$

if $c \neq 0 \rightarrow$ The symm. eqs are $\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0$

③ The vector equations of L are

$$\begin{aligned} \langle x, y, z \rangle &= \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \end{aligned}$$

Ex. Find parametric, symm & vector equations of the line through the point $A(1, 2, 3)$ & parallel to the vector $6\hat{i} - 7\hat{k}$, also find 2 pt. on the line at what pt. this line intersects the ① xy-plane ② xz-plane

param. eqs: $x = 1 + 6t$, $y = 2$, $z = 3 - 7t$

vector eqs: $\langle x, y, z \rangle = \langle 1 + 6t, 2, 3 - 7t \rangle$
 $= \langle 1, 2, 3 \rangle + \langle 6, 0, -7 \rangle t$

symm. eqs: $\frac{x-1}{6} = \frac{z-3}{-7} \Rightarrow y = 2$

Two point on line: take $t = 1 \Rightarrow x = 1 + 6 = 7$

$(7, 2, -4)$

$y = 2$

$z = 3 - 7 = -4$

Take $t = -1 \Rightarrow x = 1 - 6 = -5$

$y = 2$

$z = 3 + 7 = 10$ $(-5, 2, 10)$

xy-plane $\rightarrow z = 0$: $3 - 7t = 0$ $\left\{ \begin{array}{l} x = \frac{25}{7} \\ y = 2 \end{array} \right. \left(\frac{25}{7}, 2, 0 \right)$
 $t = \frac{3}{7}$

xz-plane $\rightarrow y = 0$ impossible

The line does not intersect the xz-plane

$$\text{let } \vec{u} \parallel L_1 \text{ \& } \vec{v} \parallel L_2 \Rightarrow \vec{u} \parallel \vec{v} \Leftrightarrow L_1 \parallel L_2$$

Ex. Determine whether the two lines L_1 & L_2 are parallel intersects or skew. If they intersects find the pt. of intersection نقطة التقاطع

$$\textcircled{1} \begin{aligned} L_1 : x &= 2-3t, y = 2t, z = 7 \\ L_2 : x &= 5+9s, y = 3-6s, z = 1 \end{aligned}$$

$$\textcircled{2} \begin{aligned} L_1 : x &= t, y = 3-t, z = 2+3t \\ L_2 : x &= 1+2s, y = 2+s, z = 5 \end{aligned}$$

$$\textcircled{3} \begin{aligned} L_1 : x &= 1+t, y = -2+3t, z = 4-t \\ L_2 : x &= 2s, y = 3+s, z = -3+4s \end{aligned}$$

$$\textcircled{1} \begin{aligned} \vec{u} &= \langle -3, 2, 0 \rangle \parallel L_1 \\ \vec{v} &= \langle 9, -6, 0 \rangle \parallel L_2 \\ \vec{u} \parallel \vec{v} &\Rightarrow L_1 \parallel L_2 \Rightarrow L_1 \& L_2 \text{ parallel} \end{aligned}$$

$$\textcircled{2} \begin{aligned} \vec{u} &= \langle 1, -1, 3 \rangle \parallel L_1 \\ \vec{v} &= \langle 2, 1, 0 \rangle \parallel L_2 \end{aligned}$$

$\vec{u} \nparallel \vec{v}$ since $|\vec{u} \times \vec{v}| \neq 0 \Rightarrow L_1 \nparallel L_2 \Rightarrow L_1 \& L_2$ not parallel
may be intersector or skewed

To check that: Assume that L_1, L_2 intersected

$$x = x \quad t = 1+2s \quad t - 2s = 1$$

$$y = y \Rightarrow 3-t = 2+s \Rightarrow -t-s = -1$$

$$z = z \quad 2+3t = 5 \quad 3t = 3$$

لدينا ٣ معادلات فختار اثنين منها لاجل لا يارد تورد والثالثة للتحقق

Ex: Find param. eqs of the line that passes through the pt. $A(1, 2, 3)$, $B(-2, 0, 1)$

$$\vec{AB} = \langle -3, -2, -2 \rangle \parallel \text{line}$$

$$\vec{AB} = \langle 3, 2, 2 \rangle$$

param. eqs. : $x = 1 + 3t$, $y = 2 + 2t$, $z = 3 + 2t$

The equation of the plane that pass through the pt $P(x_0, y_0, z_0)$ & with normal vector $\vec{n} = \langle a, b, c \rangle$
($\underline{xyz}$)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Equivalently:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

A vector equation of the plane is

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

(normal
equation)

Ex: Find the eq of the plane through the pt $(0, -6, 7)$ with normal vector $\vec{n} = \langle 2, 5, 6 \rangle$ Find the intercepts and sketch the plane

The eq of the plane $2x + 5y + 6z = 2(0) + 5(-6) + 6(7)$
 $2x + 5y + 6z = 12$

The x-intercept : $y=0, z=0$: $2x=12 \rightarrow x=6$

" y " : $x=0, z=0$: $5y=12 \rightarrow y = \frac{12}{5}$

" z " : $y=0, x=0$: $z=2$

Ex: Find the point at which the line L intersects the plane P where $L: x = 2 + 3t, y = -4t, z = 5 + t$
 $P: 4x + 5y - 2z = 18$

نحسب معادلات الخط
 ونعوض معادلات المستوي

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

$$8 + 12t - 20t - 10 - 2t = 18$$

$$-10t = 20 \rightarrow t = -2$$

نحسب قيم t في الخط
 لنصل إلى نقاط التقاطع

$$x = 2 + 3(-2) = -4$$

$$y = -4(-2) = 8$$

$$z = 5 - 2 = 3$$

$\Rightarrow$ pt. of intersection is $(-4, 8, 3)$

Ex: Find an eq. of the plane that pass through the pts $A(1, 3, 2), B(3, -1, 6), C(5, 2, 0)$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} \quad \begin{array}{l} \overrightarrow{AB} = \langle 2, -4, 4 \rangle \\ \overrightarrow{AC} = \langle 4, -1, -2 \rangle \end{array}$$

$$\vec{n} = (-4(-2) + 4)\hat{i} - (2(-2) - 4(4))\hat{j} + (2(-1) - 4(-4))\hat{k}$$

$$= 12\hat{i} + 20\hat{j} + 14\hat{k}$$

$\vec{n} \perp$ plane

$$\langle \frac{12}{2}, \frac{20}{2}, \frac{14}{2} \rangle \perp \text{plane}$$

$$\langle 6, 10, 7 \rangle \perp \text{plane}$$

الناتج دونه يوازي $\vec{n}$ وبالتالي
 لنأخذ المعادله

$$\text{The eq. of the plane: } 6x + 10y + 7z = 6(1) + 10(3) + 7(2)$$

$$= 50$$

$$6x + 10y + 7z = 50$$

Ex: Find the eq of the plane that pass through the line of intersection of the two planes $P_1: x+z=-1$ & $P_2: y=-z$ & pass through the pt $A(-3,1,1)$

$$z=0 \quad P_1 \Rightarrow x=-1$$

$$P_2 \Rightarrow y=0$$

$$B(-1,0,0)$$

on line of intersection

عندما يكون $z=0$ فإن تقاطع المستويين

الافضل هو إيجاد نقطة (x, y, z) حيث

التقاطع لهذا وذلك باخذ احد

المتغيرات الثلاثة $x/y/z$ بقيمة

وهو هنا فليكن $z=0$

$$z=1 \quad P_1 \Rightarrow x=-2$$

$$P_2 \Rightarrow y=1$$

$$C(-2,1,1)$$

ثانياً تغيير قيمة z ولتكن

القيمة السابقة $z=1$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -1 \\ 1 & 0 & 0 \end{vmatrix} = 0\hat{i} - \hat{j} + \hat{k}$$

The eq of the plane is $-y+z=0$

Ex: Find the eq of the plane of intersection of the two planes $P_1: x+y-z=1$, $P_2: 3x-3y+z=3$ and parallel to the line $x=1$, $y=3-2t$, $z=t$

$$y=0 : P_1 \Rightarrow x-z=1$$

$$P_2 \Rightarrow 3x+z=3 \quad A(1,0,0)$$

عندما يكون $y=0$

هذا التقاطع

$$y=1 : P_1 \Rightarrow x-z=0$$

$$P_2 \Rightarrow 3x+z=6 \quad B\left(\frac{3}{2}, 1, \frac{3}{2}\right)$$

$$\vec{u} = \vec{AB} = \langle \frac{1}{2}, 0, \frac{3}{2} \rangle // \text{plane}$$

افضل بـ 2

$$\vec{w} = \langle 1, 0, 3 \rangle // \text{plane}$$

$$\vec{v} = \langle 0, -2, 1 \rangle // L \rightarrow L // \text{plane}$$

$$\vec{v} = \langle 0, -2, 1 \rangle // \text{plane}$$

$$\vec{w} // \text{plane} \rightarrow \vec{v} // \text{plane}$$

$$\vec{v} \times \vec{w} \perp \text{plane}$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix} = -6\hat{i} + \hat{j} + 2\hat{k} \perp \text{plane}$$

$$\text{The eq of plane: } -6x + y + 2z = -6(1) + 0 + 2(0) \\ -6x + y + 2z = -6$$

Ex. Find the param. eqs. of the line of intersection of the two planes $P_1: x + y - 2z = 8$ & $P_2: x - y + z = 2$

$$z=0: P_1 \Rightarrow x + y = 8 \Rightarrow A(5, 3, 0) \\ P_2 \Rightarrow x - y = 2$$

$$x=0: P_1 \Rightarrow y - 2z = 8 \Rightarrow B(0, -12, 10) \\ P_2 \Rightarrow -y + z = 2$$

$$\vec{u} = \vec{AB} = \langle -5, -15, -10 \rangle // \text{line} \quad \div -5 \\ \langle 1, 3, 2 \rangle // \text{line}$$

$$\text{param. eqs} \Rightarrow x = 5 + t, y = 3 + 3t, z = 2t$$

Ex. let L be the line of intersection of the two planes

$$P_1 = x + y - 2z = 8, \quad P_2 = x - y + z = 2$$

- ① Find the eq of the plane that pass the pt $C(1, 2, 3)$ & contains the line L .

The pts. $A(5, 3, 0)$ $B(0, -12, -10)$ are on L

$\Rightarrow A(5, 3, 0)$ $B(0, -12, -10)$ $C(1, 2, 3)$ are on the plane

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & -15 & -10 \\ -4 & -1 & 3 \end{vmatrix} = (35\hat{i} + 55\hat{j} - 55\hat{k}) \div 5$$

$$-7\hat{i} + 11\hat{j} - 11\hat{k} \perp \text{plane} \Rightarrow \text{eq } -7x + 11y - 11z = -2$$

- ② Find the eq of the plane that pass the point $C(1, 2, 3)$ and perpendicular to the line L

$A(5, 3, 0)$ $B(0, -12, -10)$

$A \& B$

(نقطتين على الخط)

L هو الخط

$$\overrightarrow{AB} \parallel L \Rightarrow \langle -5, -15, -10 \rangle \parallel L$$

$AB \parallel L$

$$L \perp \text{plane} \Rightarrow \langle -5, -15, -10 \rangle \perp \text{plane} \quad \div 5$$

$$\langle 1, 3, 2 \rangle$$

$$\text{eq of plane} : x + 3y + z = 10$$

استخدمنا C ولا حاجة استخدام A و B
لأنه إذا كان الخط L موازيا لخط AB فإنه يحدد المستوي المطلوب

Let P_1 & P_2 be two planes such that $\vec{n}_1 \perp \vec{p}_1$ and $\vec{n}_2 \perp \vec{p}_2$ then

- 1) P_1 parallel to P_2 , $P_1 \parallel P_2 \Leftrightarrow \vec{n}_1 \parallel \vec{n}_2$
- 2) If P_1, P_2 are not parallel & θ is the angle between P_1, P_2 then $\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$

3) $P_1 \perp P_2 \Leftrightarrow \vec{n}_1 \cdot \vec{n}_2 = 0$

Ex: Find the angle between $P_1: x-y=3$ & $P_2: x+2y+z=1$

$$\vec{n}_1 = \langle 1, -1, 0 \rangle$$

$$\vec{n}_2 = \langle 1, 2, 1 \rangle$$

$$\cos \theta = \frac{1 - 2 - 0}{\sqrt{2} \sqrt{6}} = \frac{-1}{\sqrt{12}}$$

$$\theta = \cos^{-1} \frac{-1}{\sqrt{12}} = 106.7^\circ$$

Ex: show that the two planes are orthogonal

$$P_1: 2x - 3y + z = 0 \quad P_2: x + 2y + 4z = 3$$

$$\vec{n}_1 = \langle 2, -3, 1 \rangle \perp P_1 \quad \& \quad \vec{n}_2 = \langle 1, 2, 4 \rangle \perp P_2$$

$$\vec{n}_1 \cdot \vec{n}_2 = 2(1) + -3(2) + 1(4) = 0 \rightarrow \vec{n}_1 \perp \vec{n}_2 \Rightarrow P_1 \perp P_2$$

Ex: Find a that makes P_1 & P_2 parallel

$$P_1: 2x + 3ay - 2z = 1 \quad P_2: \frac{3}{2}ax + 9y - 3z = 0$$

$$\vec{n}_1 = \langle 2, 3a, -2 \rangle \perp P_1$$

$$\vec{n}_2 = \langle \frac{3}{2}a, 9, -3 \rangle \perp P_2$$

$$\frac{2}{\frac{3}{2}a} = \frac{3a}{9} = \frac{-2}{-3}$$

$$P_1 \parallel P_2 \Rightarrow \vec{n}_1 \parallel \vec{n}_2$$

$$\frac{a}{\frac{3}{2}} = \frac{2}{3}$$

$$a = 2$$

The distance from the pt $A(x_0, y_0, z_0)$ to the plane $P: ax + by + cz = d$ is

$$\text{Dist} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Ex: Find the distance from the pt $A(-1, 2, 3)$ and the plane $P: 2x - 4y + z = 1$

$$P: 2x - 4y + z - 1 = 0 \quad \vec{n} = \langle 2, -4, 1 \rangle$$

$$\text{dist} = \frac{|-8|}{\sqrt{21}} = \frac{8}{\sqrt{21}}$$

Let P_1, P_2 be two planes $n_1 \perp P_1$ & $n_2 \perp P_2$

- ① If $P_1 \times P_2 \Rightarrow$ the dist between P_1 & $P_2 = 0$
- ② If $P_1 \not\parallel P_2 \Rightarrow$ the dist between P_1 & P_2 is

$\text{Dist}(P_1, P_2) = \text{Dis}(A, P_2)$ where A is a point on P_1

Ex: Find the distance between the planes P_1, P_2 .

يجب ان نرى ان تكون المستويان اذا متوازيان او يتقاطعا

① $P_1: x - 2y + z = 3$, $P_2: 2x - y + 2z = 6$

② $P_1: 10x + 2y - 2z = 5$, $P_2: 5x + y - z = 1$

① $\vec{n}_1 = \langle 1, -2, 1 \rangle \perp P_1$ & $\vec{n}_2 = \langle 2, -1, 2 \rangle \perp P_2$

$\vec{n}_1 \parallel \vec{n}_2$ since $|\vec{n}_1 \times \vec{n}_2| \neq 0 \Rightarrow P_1 \times P_2$
 P_1, P_2 intersected

Dist = 0

② $\vec{n}_1 = \langle 10, 2, -2 \rangle \perp P_1$ & $\vec{n}_2 = \langle 5, 1, -1 \rangle \perp P_2$

$\vec{n}_1 \parallel \vec{n}_2$ since $2\vec{n}_2 = \vec{n}_1 \Rightarrow P_1 \parallel P_2$

To find the distance : we find a pt on P_1 :

$P_1: 10x + 2y - 2z = 5$ take $x=0, y=0 \rightarrow z = -\frac{5}{2}$

$A(0, 0, -\frac{5}{2})$ on P_1

$Dis(P_1, P_2) = Dist(A, P_2) = \frac{1}{2\sqrt{3}}$

المسافة

12.6) Cylinders and Quadric Surfaces

Def: Cylinders are surfaces that results by moving a curve in a direction of a fixed axis (line)

Ex: ① $z = x^2$, $x^2 + y^2 = 4$ are (cylinders)
 ② All planes are cylinders: $x - 2y = 1$, $z = 3$ are cylinders
 ③ $x^2 - 3y + z = 5$, $x + y \cos z$ are not cylinders

السطح التربيعي

Def: A Quadric surface is a graph of a second degree eq. in the variable x, y, z in the form.

احد من الرموز صريحة او خفية وضربيه

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

وهذه المعادلة اعلى اى لها (2) ولا يظهر فيها المتغيرات او الجذور او الاكسور

where $A, B, C, \dots, J$ are scalars & not all $= 0$

(مركز التقاطع)

Ex: ① Give the name of the trace of the quadric surface

$$2x^2 + y^2 - z^2 = 16$$

(a) in the plane $z = 1$

(b) in the plane $y = 1$

(a) $z = 1$: $2x^2 + y^2 - 1 = 16 \Rightarrow 2x^2 + y^2 = 17 \Rightarrow \frac{2x^2}{17} + \frac{y^2}{17} = 1$ Ellipse

(b) $y = 1$: $2x^2 + 1 - z^2 = 16 \Rightarrow 2x^2 - z^2 = 15 \Rightarrow$ Hyperbola

② Identify the trace of the surface $x^2 + y^2 + z = 10$

(a) in the plane $z = 1$

(b) in the plane $z = 2$

(a) $z = 1$: $x^2 + y^2 + 1 = 10 \Rightarrow x^2 + y^2 = 9 \Rightarrow$ circle

(b) $z = 2$: $x^2 + y^2 + 2 = 10 \Rightarrow x^2 + y^2 = 8 \Rightarrow$ circle

(c) $x = 2$: $4 + y^2 + z = 10 \Rightarrow y^2 + z = 6 \Rightarrow z = 6 - y^2 \Rightarrow$ Parabola

Graphs of quadric surfaces

شكل السطح

① Ellipsoid

① معطى المعادلات في الصورة التالية

equation $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

All traces are ellipsoid if $a=b=c$, the ellipsoid is a sphere

② Cone

شكل

equation $\Rightarrow \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Horizontal traces are ellipses vertical traces in the planes $x=k$ & $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$

③ Elliptic Paraboloid

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ شكل السطح

Horizontal traces are ellipses vertical traces are parabolas
The variable raised to the first power indicates the axis of the paraboloid

Ex: use traces to sketch the surface

التقاطع ليس مع المحاور
 $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ show the intercepts & give the name
 المحاور y -axis

surface name is ellipsoid Intercepts

المحور z والمحور x

x -intercepts, $y=0, z=0$

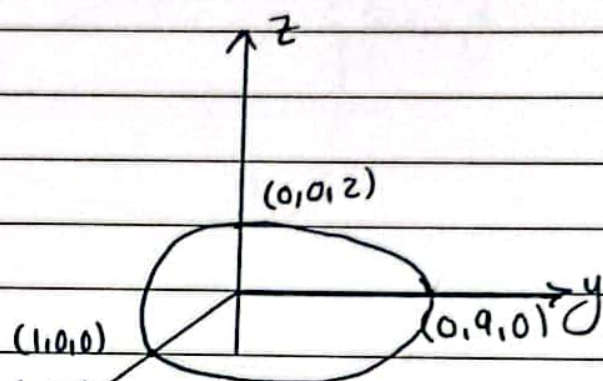
$$x^2 + 0 + 0 = 1 \Rightarrow x = \pm 1$$

y -intercepts, $x=0, z=0$

$$0 + \frac{y^2}{9} + 0 = 1 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$

z -intercepts, $y=0, x=0$

$$0 + 0 + \frac{z^2}{4} = 1 \Rightarrow z^2 = 4 \Rightarrow z = \pm 2$$



Ex: Classify & sketch the surface $x^2 + 2z^2 - 6x - y + 10 = 0$

surface name is elliptic paraboloid

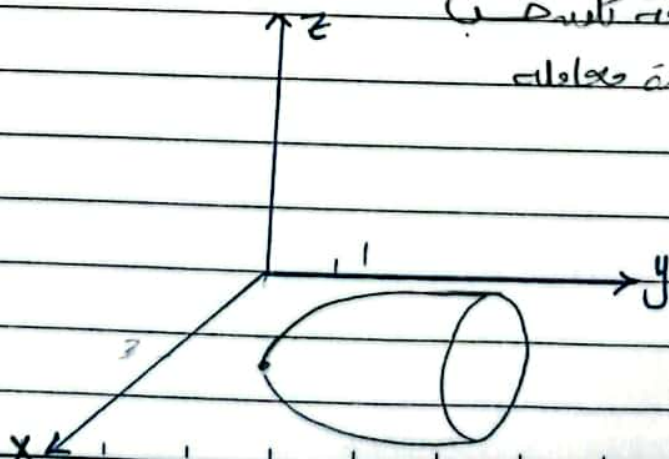
يجب كمال للمرجح

$$\begin{aligned} x^2 - 6x + 2z^2 &= y - 10 \\ x^2 - 6x + 9 + 2z^2 &= y - 1 \\ (x-3)^2 + 2z^2 &= y - 1 \end{aligned}$$

المحور y المحاور x والمحور z
 التي ليس عليه تجميع
 والفتحة تكون ب
 اتجاه y

vertex (3, 1, 0)

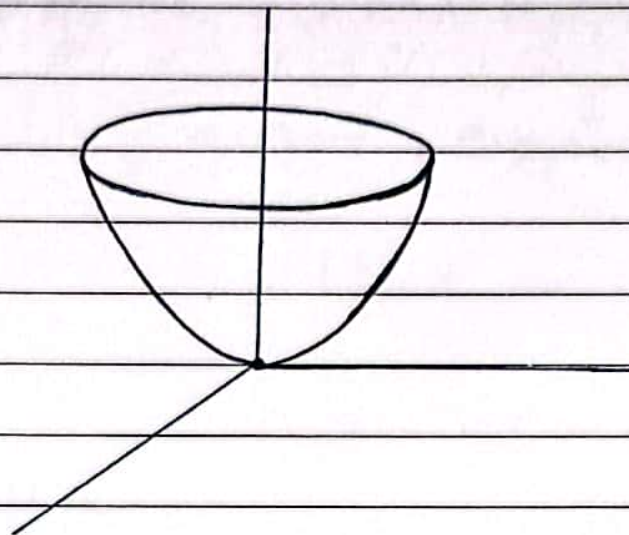
نصف المحاور



Ex. Identify & sketch the surfaces

① $z = 4x^2 + y^2 \rightarrow$ Elliptic Paraboloid

vertex = $(0, 0, 0)$



② $\frac{x^2}{4} + y^2 - 6y + \frac{z^2}{3} = 0$

مركز القوس
في الموضع
الأسفل

$$\frac{x^2}{4} + (y-3)^2 + \frac{z^2}{3} = 9$$

center $(0, 3, 0)$

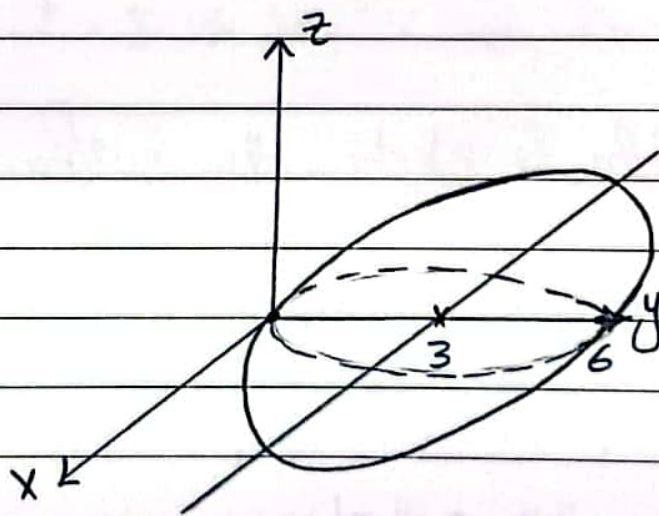
y-intercept

$$x=0, z=0$$

$$y^2 - 6y = 0$$

$$y(y-6) = 0$$

$$y=0 \quad y=6$$



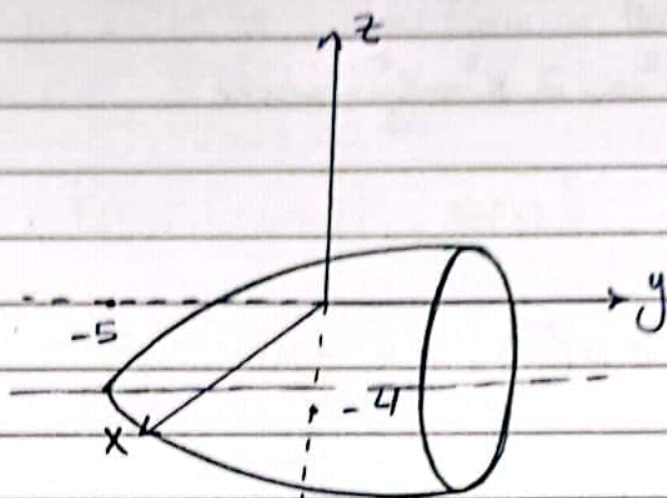
*4
 (3) $\left(\frac{x^2}{4} - y + \frac{z^2}{4} + 2z = 1\right)$, E.P

$$x^2 + z^2 + 8z = 4 + 4y$$

$$x^2 + (z+4)^2 = 4y + 20$$

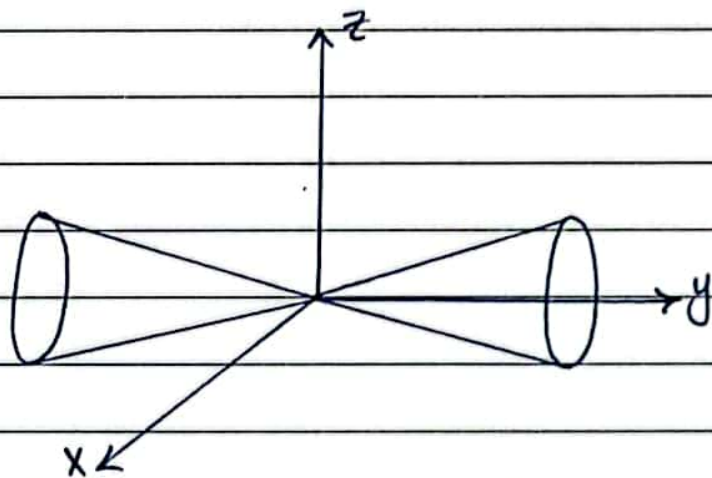
vertex = $\begin{pmatrix} 0, -4, -5 \end{pmatrix}$
 $= (0, -5, -4)$

z-intercept
 $x=0, y=0$

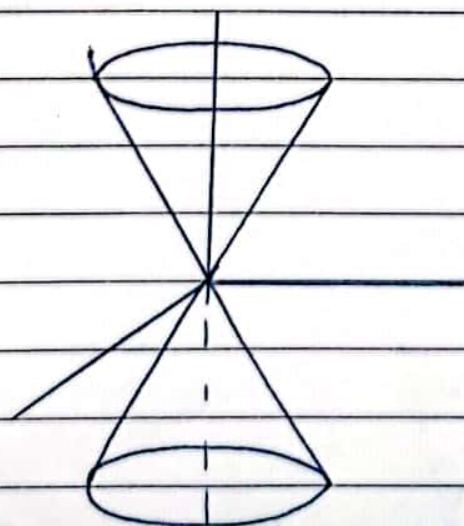


(4) $4x^2 + 2z^2 = y^2$ → Cone

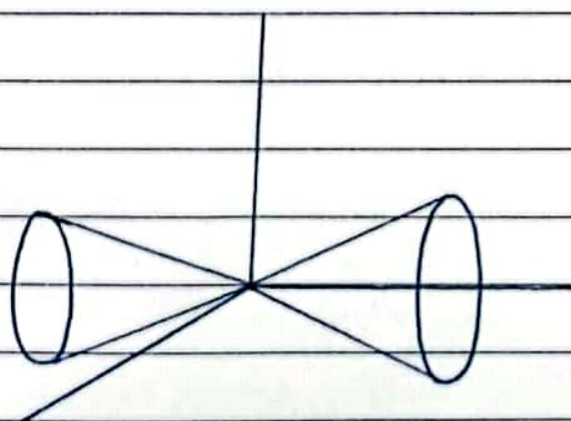
vertex = $(0, 0, 0)$



(5) $z^2 = 2x^2 + y^2$ → cone



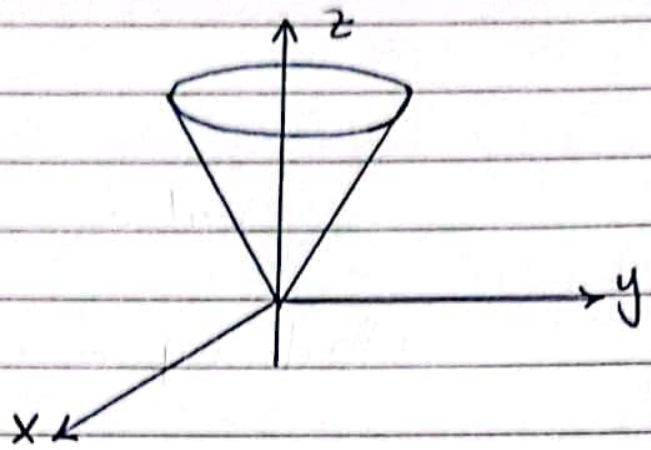
(6) $y^2 = x^2 + 4z^2$, cone



نبيع الشيفر

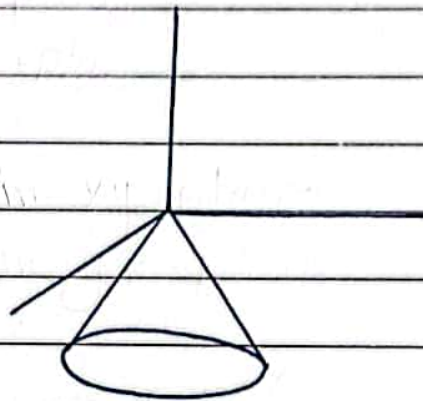
⑦ $z = \sqrt{2x^2 + y^2}$

$z^2 = 2x^2 + y^2 \rightarrow \text{cone}$



⑧ $z = \ominus \sqrt{2x^2 + y^2}$

cone

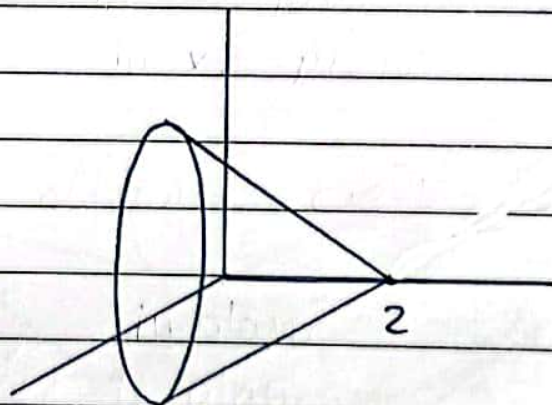


⑨ $2-y = \sqrt{2x^2 + z^2} \rightarrow \text{cone}$

$(2-y)^2 = 2x^2 + z^2$

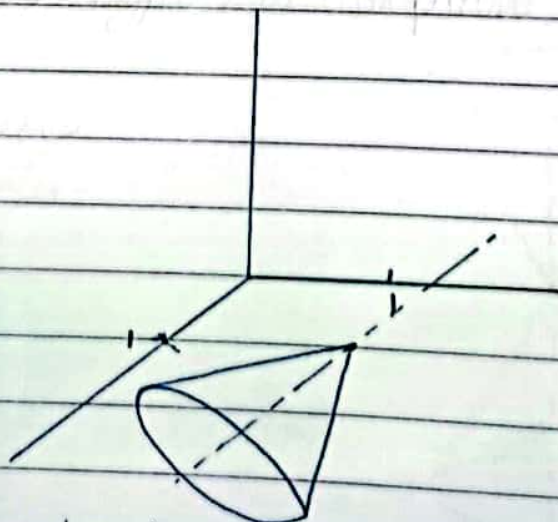
لنغير الاتجاه فنكتب الشيفر

vertex = (0, 2, 0)



⑩ $x-1 = \sqrt{(y-1)^2 + z^2} \rightarrow \text{cone}$

vertex = (1, 1, 0)



13.1) Vectors Function and vector curves

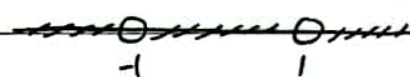
A vector function denoted by $\vec{r}(t)$ is a function in the variable (t) with domain $A \subset \mathbb{R}$ & its range is a set of vectors

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \quad (t \in A)$$

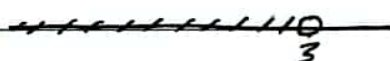
$$\text{Dom}(\vec{r}) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

Ex. Find the domain of $\vec{r}(t) = \langle \frac{t}{t^2-1}, \ln(3-t), \sqrt{t} \rangle$

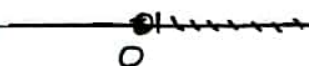
$$\frac{t}{t^2-1} : t^2-1 \neq 0 \rightarrow t \neq \pm 1$$



$$\ln(3-t) : 3-t > 0 \rightarrow t < 3$$



$$\sqrt{t} : t \geq 0$$

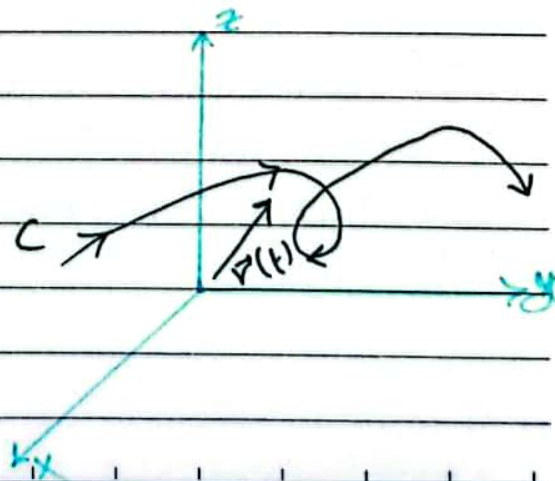


$$\text{Dom}(\vec{r}) = [0, 1) \cup (1, 3)$$

Geometrically 3

The vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ defined a vector curve C traced out by the tip of the moving vector $\vec{r}(t)$.

The direction of C is at the direction of the moving tip when t increases as shown in the Figure. The vector $\vec{r}(t)$ is called a position vector.



Ex. sketch & describe the curve defined by the vcc. Fun.

① $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$

② $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$

③ $\vec{r}(t) = \langle \sin t, t \rangle$

$t=0 \rightarrow (1, 2, -1) \quad t=1 \rightarrow (2, 7, 5)$

① $x=1+t, y=2+5t, z=-1+6t$ It's a line
(1, 2, -1)

② $x=\cos t, y=\sin t, z=t$

$\cos^2 t + \sin^2 t = 1 \Rightarrow x^2 + y^2 = 1, z=t$ circular helix
زبر (دائري)

③ $x=\sin t, y=t \Rightarrow x=\sin y$

$\vec{r}(t) = \langle 3\cos t - 1, 5 - \sin t, 3t \rangle$

$x = 3\cos t - 1$
 $y = 5 - \sin t$
 $z = 3t$

$\cos t = \frac{x+1}{3}$
 $\sin t = 5-y$

$\cos^2 t + \sin^2 t = 1$
 $\frac{(x+1)^2}{9} + (5-y)^2 = 1$
 $z = 3t$

$t=0 \rightarrow (2, 5, 0)$

IF $a, b, c > 0$ the vector function: $\vec{r}(t) = \langle a\cos t, b\sin t, ct \rangle$ is called a helix when $a=b$ then this vector function is called a circular helix

Let $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$ & $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

① IF $\lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t)$ all exist then

$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$

② If at least one of the limits does not exist then $\lim_{t \rightarrow a} \vec{r}(t)$ does not exist (DNE)

Ex: If $\vec{r}(t) = \left\langle \frac{t^2}{\tan(2t)}, \frac{t-1}{t^2-1}, \frac{\sin \pi t}{\ln(t+1)} \right\rangle$ Find $\lim_{t \rightarrow 0} \vec{r}(t)$

$$\lim_{t \rightarrow 0} \frac{t^2}{\tan(2t)} = \lim_{t \rightarrow 0} \frac{2t}{2 \sec^2(2t)} = \frac{0}{2} = 0 \quad \text{L'Hôpital}$$

$$\lim_{t \rightarrow 0} \frac{t-1}{t^2-1} = \frac{-1}{-1} = 1$$

$$\lim_{t \rightarrow 0} \frac{\sin \pi t}{\ln(t+1)} = \lim_{t \rightarrow 0} \frac{\pi \cos \pi t}{\frac{1}{t+1}} = \pi$$

$$\lim_{t \rightarrow 0} \vec{r}(t) = \langle 0, 1, \pi \rangle$$

Ex: Find $\lim_{t \rightarrow \infty} \vec{r}(t)$ where $\vec{r}(t) = \left\langle t e^{-t}, \frac{t^3+t}{2t^3-1}, t \sin\left(\frac{1}{t}\right) \right\rangle$

$$\lim_{t \rightarrow \infty} t e^{-t} = \lim_{t \rightarrow \infty} \frac{t}{e^t} = \lim_{t \rightarrow \infty} \frac{1}{e^t} = \frac{1}{\infty} = 0$$

$$\lim_{t \rightarrow \infty} \frac{t^3+t}{2t^3-1} = \lim_{t \rightarrow \infty} \frac{t^3}{2t^3} = \frac{1}{2}$$

$$\lim_{t \rightarrow \infty} t \sin \frac{1}{t} = \lim_{t \rightarrow \infty} \frac{\sin \frac{1}{t}}{1/t} = \lim_{t \rightarrow \infty} \frac{(\cos \frac{1}{t})(-\frac{1}{t^2})}{-\frac{1}{t^2}} = \lim_{t \rightarrow \infty} \cos \frac{1}{t} = 1$$

$$\therefore \lim_{t \rightarrow \infty} \vec{r}(t) = \left\langle 0, \frac{1}{2}, 1 \right\rangle$$

Ex: $\lim_{t \rightarrow 0^-} \langle \sqrt{t}, \sin t \rangle$ DNE since $\lim_{t \rightarrow 0^-} \sqrt{t}$ DNE

Let $a \in \text{Dom } \vec{r}(t)$. Then $\vec{r}(t)$ is cont. at $\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$

Ex. $\vec{r}(t) = \left\langle \frac{t^2}{\tan(2t)}, \frac{t-1}{t^2-1}, \frac{\sin \pi t}{\ln(t+1)} \right\rangle$ is cont on $\text{Dom}(\vec{r})$

$$\tan 2t \neq 0$$

$$2t \neq 0$$

$$2t \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$$

$$t^2 - 1 \neq 0$$

$$t \neq \pm 1$$

$$t+1 > 0 \rightarrow t > -1$$

$$t+1 \neq 1 \rightarrow t \neq 0$$

$$\text{Dom}(\vec{r}) = (-1, \infty) \setminus \left\{ 0, 1, -\frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots, \frac{\pi}{4}, \frac{3\pi}{2}, \frac{5\pi}{4}, \dots \right\}$$

$\therefore \vec{r}$ conts on $\text{Dom}(\vec{r})$

13.2) Derivative & Integral of vector Functions

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a smooth curve then

$$\frac{d\vec{r}}{dt} = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \text{ if the lim exist } \frac{d\vec{r}}{dt} = \vec{r}'(t)$$

$$\# \text{ IF } \vec{r}(t) = \langle f, g, h \rangle \Rightarrow \vec{r}' = \langle f', g', h' \rangle$$

Ex: IF $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin 2t \rangle$

$$\vec{r}'(t) = \langle 3t^2, t(-e^{-t}) + e^{-t}, 2\cos 2t \rangle$$

Geometrically 4.

Let C be the curve of the vector function $\vec{r}(t)$ & A is a pt on C
 Then $\vec{r}'$ is a vector tangent to the tangent line to the vector curve C that pass through the pt A

The unit tangent vector to the curve C is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Ex. Find the unit tangent vector of $\vec{r}(t) = (1+t^3)\hat{i} + te^{-t}\hat{j}$ when $t=0$ & at the pt $A(1,0)$

$$\vec{r}'(t) = \langle 3t^2, (1-t)e^{-t} \rangle$$

$$\vec{r}'(0) = \langle 0, 1 \rangle \Rightarrow \vec{T} = \frac{\langle 0, 1 \rangle}{|\langle 0, 1 \rangle|} = \langle 0, 1 \rangle$$

At $A(1,0)$: $x=1, y=0$: $x=1+t^3, y=te^{-t}$

$$1+t^3=1 \Rightarrow t=0 \quad \vec{r}'(t)|_A = \vec{r}'(0) = \langle 0, 1 \rangle \quad \vec{T} = \langle 0, 1 \rangle$$

$te^{-t}=0$

Ex. let $\vec{r}(t) = (1+t^3)\hat{i} + te^{-t}\hat{j} + \sin(2t)\hat{k}$

① Find a tangent vector of $\vec{r}(t)$

② Find a unit tangent vector to $\vec{r}(t)$ at $t=0$

③ Find a unit tangent vector to $\vec{r}(t)$ at the pt $A(1,0,0)$

$$\textcircled{1} \vec{r}'(t) = 3t^2\hat{i} + (1-t)e^{-t}\hat{j} + 2\cos 2t\hat{k}$$

$$\textcircled{2} \vec{r}'(0) = 0\hat{i} + \hat{j} + 2\hat{k} \Rightarrow \vec{T}(0) = \frac{\hat{j} + 2\hat{k}}{\sqrt{1+4}} = \frac{\hat{j} + 2\hat{k}}{\sqrt{5}}$$

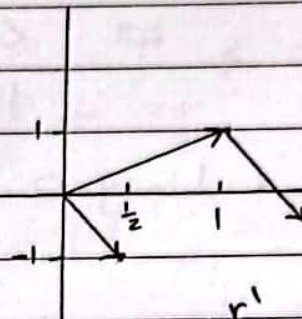
$$\textcircled{3} (1,0,0) \rightarrow x=1, y=0, z=0$$

$$\begin{aligned} x=1+t^3 &\rightarrow 1=1+t^3 \rightarrow t=0 \\ y=te^{-t} &\rightarrow te^{-t}=0 \\ z=\sin 2t &\rightarrow \sin 2t=0 \end{aligned} \quad \left. \vphantom{\begin{aligned} x=1+t^3 \\ y=te^{-t} \\ z=\sin 2t \end{aligned}} \right\} \vec{r}'(0) = \hat{j} + 2\hat{k} \rightarrow \vec{T}(0) = \frac{\hat{j} + 2\hat{k}}{\sqrt{5}}$$

Ex. Let $\vec{r}(t) = \sqrt{t} \hat{i} + (2-t) \hat{j}$ Find $\vec{r}'(t)$ & sketch the position vector $\vec{r}'(1)$ & $\vec{r}''(1)$

$$\Rightarrow \vec{r}'(t) = \frac{1}{2\sqrt{t}} \hat{i} - \hat{j} \rightarrow \vec{r}'(1) = \frac{1}{2} \hat{i} - \hat{j} \quad \left(\frac{1}{2}, -1\right)$$

$$\Rightarrow \vec{r}(1) = \hat{i} + \hat{j} \quad (1, 1)$$



Ex. Find parametric eqs. of the (tangent line) to the helix $x = 2\cos t$, $y = \sin t$, $z = t$ at the pt. $A(0, 1, \frac{\pi}{2})$

$$\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$$

$$\text{At } A: x=0 \rightarrow 2\cos t=0$$

$$\vec{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle$$

$$y=1 \rightarrow \sin t=1$$

$$z=\frac{\pi}{2} \rightarrow t=\frac{\pi}{2}$$

$$\vec{r}'\left(\frac{\pi}{2}\right) = \langle -2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1 \rangle = \langle -2, 0, 1 \rangle$$

$\vec{r}' \parallel$ tangent line

parametric eqs: $x = 0 - 2t$, $y = 1 + 0t$, $z = \frac{\pi}{2} + t$

$$x = -2t, \quad y = 1, \quad z = \frac{\pi}{2} + t$$

Let $\vec{u}(t)$, $\vec{v}(t)$ be vector functions & $f(t)$ function & a, b scalar

$$\textcircled{1} \frac{d}{dt} (a\vec{u} \pm b\vec{v}) = a \frac{d\vec{u}}{dt} \pm b \frac{d\vec{v}}{dt}$$

$$\textcircled{2} \frac{d}{dt} (f\vec{u}) = f \frac{d\vec{u}}{dt} + f' \vec{u}$$

$$\textcircled{5} \frac{d}{dt} \vec{u}(f(t)) =$$

$$\textcircled{3} \frac{d}{dt} (\vec{u} \cdot \vec{v}) = \vec{u} \frac{d\vec{v}}{dt} + \vec{v} \frac{d\vec{u}}{dt}$$

$$\textcircled{4} \frac{d}{dt} (\vec{u} \times \vec{v}) = \vec{u} \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v} = \vec{u}'(t) \times \vec{v} + \vec{u} \times \vec{v}'(t)$$

Ex: Let $\vec{u} = \langle t, e^{-t}, \sin 2t \rangle$, $\vec{v} = \langle t-2, t^2+2, e^{2t} \rangle$

Then $\frac{d}{dt}(2\vec{u} - 3\vec{v})$

$$2 \frac{d\vec{u}}{dt} - 3 \frac{d\vec{v}}{dt} = 2 \langle 1, -e^{-t}, 2\cos 2t \rangle - 3 \langle 1, 2t, 2e^{2t} \rangle$$

$$= \langle -1, -2e^{-t} - 6t, 4\cos t - 6e^{2t} \rangle$$

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int \vec{r}(t) dt = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle$$

Ex: $\vec{r}(t) = 3\cos t \hat{i} + \sin t \hat{j} + 2t \hat{k}$ Find

① $\int \vec{r}(t) dt$

$$\int \vec{r}(t) dt = \langle \int 3\cos t dt, \int \sin t dt, \int 2t dt \rangle$$

$$= \langle 3\sin t + C_1, -\cos t + C_2, t^2 + C_3 \rangle$$

$$= \langle 3\sin t, -\cos t, t^2 \rangle + \vec{C}$$

② $\int_0^{\pi/2} \vec{r}(t) dt$

$$\int_0^{\pi/2} \vec{r}(t) dt = \langle \int_0^{\pi/2} 3\cos t dt, \int_0^{\pi/2} \sin t dt, \int_0^{\pi/2} 2t dt \rangle$$

$$= \langle 3, 1, \frac{\pi^2}{4} \rangle$$

13.3) Arc length & Curvature

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$

① The arc length of $\vec{r}(t)$ from $t=a$ to $t=b$ is

$$L = \int_a^b |\vec{r}'| dt$$

② ~~The arc length function of $\vec{r}(t)$ in the direction of increasing t is~~

$$s(t) = \int_a^t |\vec{r}'| dt$$

③ ~~Parametrization of $\vec{r}(t)$ with respect to the arc length function is $\vec{r}(t(s))$~~

Ex. let $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$

④ Find the arc length of $\vec{r}(t)$ from the pt $A(1,0,0)$ to $B(1,0,2\pi)$

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$A(1,0,0): \begin{cases} \cos t = 1 \\ \sin t = 0 \\ t = 0 \end{cases} \quad t=0$$

$$B(1,0,2\pi): \begin{cases} \cos t = 1 \\ \sin t = 0 \\ t = 2\pi \end{cases} \quad t=2\pi$$

$$0 \leq t \leq 2\pi$$

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + \cos^2 t + 1} dt = \int_0^{2\pi} \sqrt{2} dt = \sqrt{2}(2\pi - 0) = 2\sqrt{2}\pi$$

The curvature of a curve C , written as $K(t)$ is the magnitude of the rate of change of the unit tangent vector with respect to the arc length. Also, It is a measure of how quickly the curve changes direction at a pt.

let C be a curve & let $\vec{r}(t)$ be its vector function

Then ① $K(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$ ✓

② $K(t) = \frac{|\vec{T}'|}{|\vec{r}'|}$

③ If C is a plane curve and $\vec{r}(t) = x\hat{i} + f(x)\hat{j}$

then $K(t) = \frac{|f''|}{(1+(f')^2)^{\frac{3}{2}}}$

Ex: Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

① at a general point

$$\vec{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \vec{r}'' = \langle 0, 2, 6t \rangle$$
$$|\vec{r}'|^2 = 1 + 4t^2 + 9t^4 \quad \Rightarrow \quad |\vec{r}''|^2 = 0 + 4 + 36t^2$$

$$K(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{\sqrt{(1+4t^2+9t^4)(4+36t^2) - (4t+18t^3)^2}}{(1+4t^2+9t^4)^{\frac{3}{2}}}$$

② at the origin pt $(0,0,0)$ $t=0, t^2=0, t^3=0 \rightarrow t=0$

$$K(0) = \frac{\sqrt{1(4) - 0^2}}{1^{\frac{3}{2}}} = 2$$

Ex: Find the curvature of $y=x^2$ at the pt $(2,4)$

$$x=t, \quad y=t^2$$
$$\vec{r} = \langle t, t^2 \rangle, \quad r' = \langle 1, 2t \rangle, \quad r'' = \langle 0, 2 \rangle$$

$$\text{pt. } (2,4) \Rightarrow x=t \rightarrow t=2 \Rightarrow r' = \langle 1, 4 \rangle, \quad r'' = \langle 0, 2 \rangle$$

$$K(2) = \frac{\sqrt{17(4) - 8^2}}{(17)^{\frac{3}{2}}} = \frac{\sqrt{68 - 64}}{17\sqrt{17}} = \frac{2}{17\sqrt{17}} = \frac{2\sqrt{17}}{289}$$

Ex: Show that the curvature of a circle of radius a is $\frac{1}{a}$

Consider the circle $x^2 + y^2 = a^2$ in 2D

$$x = a \cos t, \quad y = a \sin t$$

$$\vec{r}(t) = \langle a \cos t, a \sin t \rangle$$

$$|r'| = a$$

$$r'(t) = \langle -a \sin t, a \cos t \rangle$$

$$|r''| = a$$

$$r''(t) = \langle -a \cos t, -a \sin t \rangle$$

$$K(t) = \frac{|r' \times r''|}{|r'|^2} = \frac{1}{a} \quad \#$$

let $\vec{r}(t)$ be a smooth curve

① The unit normal vector $\vec{N}(t) = \frac{T'}{|T'|}$

② The Binormal vector $\vec{B}(t) = \vec{T} \times \vec{N}$

Ex: Find the normal & Binormal vectors for $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$

A vector equation for the line segment that joins $P(a, b, c)$ to $Q(d, e, f)$ is $\vec{r}(t) = \langle tP + (1-t)Q \rangle$, $0 \leq t \leq 1$

Ex: Find a vector eq. & paramm. eq for the line segment that joins $P(1, 0, 1)$ $Q(2, 3, 1)$

$$\begin{aligned}\vec{r}(t) &= \langle tP, (1-t)Q \rangle \\ &= \langle t(1) + (1-t)2, t(0) + (1-t)3, t(1) + (1-t)(1) \rangle \\ &= \langle 2-t, 3-3t, 1 \rangle \quad 0 \leq t \leq 1\end{aligned}$$

parammetric eq: $x = 2-t, y = 3-3t, z = 1$

14.1) Functions of several variables

$$D = \{(x, y) \in \mathbb{R}^2 : f(x, y), (x, y) \in \}$$

$$R = \{z \in \mathbb{R} : z = f(x, y), (x, y) \in D\}$$

Ex: let $f(x, y) = x + \ln(y^2 - x)$ Then find $f(3, 2)$

$$f(3, 2) = 3 + \ln(4 - 3) = 3 \rightarrow \text{range}$$

Ex: Find & sketch the domain of the function.

$$(1) f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$$

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : x+y+1 \geq 0, x \neq 1\}$$

$$x+y+1=0$$

$$x+y=-1$$

$$x=0 \rightarrow y=-1$$

$$x=2 \rightarrow y=-2$$

لا بد من فصل

لأنه يجب

مساواة

المجموعة منطوق

اقل نصوص نقطة

اختبار في المطابقة

$$(2) f(x, y) = \ln(y^2 - x)$$

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : y^2 - x > 0\}$$

$$\{(x, y) \in \mathbb{R}^2 : y^2 > x\}$$

$$y^2 = x$$

لا بد من فصل
وجوه مساواة

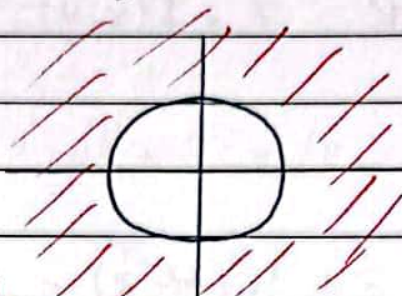
$$(3) f(x, y) = \sqrt{x^2 + y^2 - 9}$$

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 9 \geq 0\}$$

$$= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 9\}$$

$$x^2 + y^2 = 9$$

circle

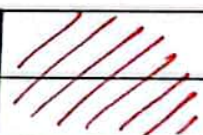


بالدائرة إذا كانت لطبيعية البر أكبر منطقة المجال
خارج الدائرة أما إذا كانت أصغر تكون منطقة المجال داخل الدائرة

$$(4) f(x, y) = \sqrt{xy}$$

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$$

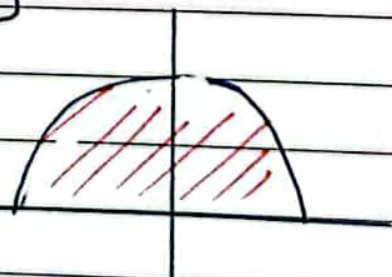
$xy \geq 0 \rightarrow$ If $x \geq 0$ & $y \geq 0$, First quadrant
 $x \leq 0$ & $y \leq 0$, third quadrant



$$(5) f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$$

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : y \geq 0, x^2 + y^2 \leq 25\}$$

أصغر
فالمجموعة داخل
الدائرة



$$(6) f(x, y) = 3x$$

$$f(x, y) = 3x + 0y$$

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2\} = \mathbb{R}$$

Ex: Find the domain and range of the function

$$z$$
$$(f(x,y)) = 2 - 3\sqrt{9 - x^2 - y^2}$$

المجال التربيعي
طالعانية التربيعية

$$\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$$

$$\textcircled{1} \quad \sqrt{9 - x^2 - y^2} \geq 0 \Rightarrow -3\sqrt{9 - x^2 - y^2} \leq 0$$

$$2 - 3\sqrt{9 - x^2 - y^2} \leq 2 \Rightarrow z \leq 2$$

$$\textcircled{2} \quad x^2 + y^2 \geq 0 \Rightarrow -x^2 - y^2 \leq 0 \Rightarrow 9 - x^2 - y^2 \leq 9$$

$$\sqrt{9 - x^2 - y^2} \leq \sqrt{9} \Rightarrow -3\sqrt{9 - x^2 - y^2} \geq -3\sqrt{9}$$

$$2 - 3\sqrt{9 - x^2 - y^2} \geq -3\sqrt{9} + 2 \Rightarrow 2 - 3\sqrt{9 - x^2 - y^2} \geq -7$$
$$z \geq -7$$

$$\therefore \text{range} = [-7, 2]$$

يجب ان
تكون جميع
المتغيرات
موجبة

Ex: Find the domain & range of the function, $f(x,y) = x^2 + 2y^2$

$$\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2\} = \mathbb{R}^2$$

$$x^2 + 2y^2 \geq 0 \Rightarrow z \geq 0$$

$$\therefore \text{range}(f) = [0, \infty)$$

$$f(x,y) = x^2 - 2y^2$$

$$x^2 - 2y^2 \geq 0 \quad \times$$

$$\therefore \text{Range} = \mathbb{R}$$

If P is a function of two variables with D then the graph of P is the set of all points (x, y, z)

Ex: sketch the graph of the function

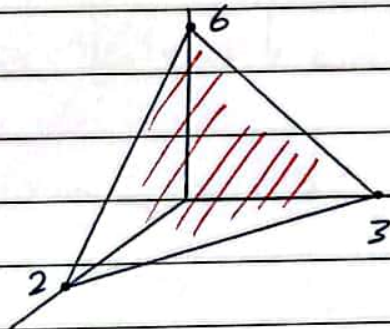
① $P(x, y) = 6 - 3x - 2y$

$z = 6 - 3x - 2y$ → 3 variables → plane

x-intercept, $y = z = 0$

$$3x = 6$$

$$x = 2$$

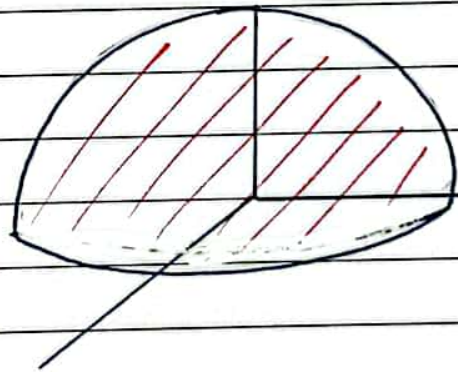


② $P(x, y) = \sqrt{9 - x^2 - y^2}$

$$z = \sqrt{9 - x^2 - y^2}, \quad \underline{z \geq 0}$$

$$z^2 = 9 - x^2 - y^2$$

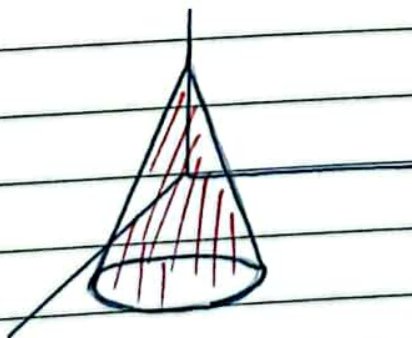
$$x^2 + y^2 + z^2 = 9 \rightarrow \text{hemisphere}$$



③ $z = f(x, y) = 6 - \sqrt{x^2 + 2y^2}$

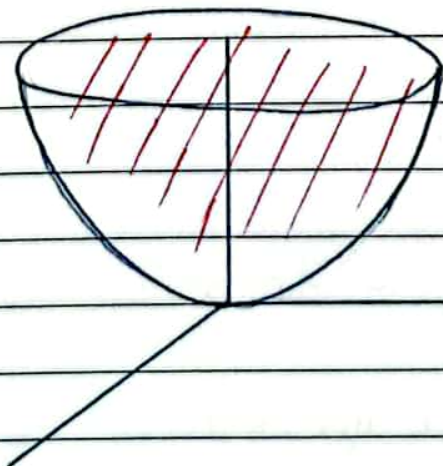
$$z = 6 - \sqrt{x^2 + 2y^2}$$

$$(z - 6)^2 = x^2 + 2y^2 \rightarrow \text{cone}$$



④ $P(x, y) = x^2 + 2y^2$

$$z = x^2 + 2y^2 \rightarrow \text{paraboloid}$$



Describe how the graph of the function g can be obtained from the graph of the function f in each case:

① $g(x, y) = f(x, y) + 2 \Rightarrow$ shift the graph of f upward 2 units

② $g(x, y) = 2f(x, y) \Rightarrow$ stretch the graph of f vertically by a factor of 2 units

③ $g(x, y) = -f(x, y)$

Reflect the graph of f about the xy -plane

④ $g(x, y) = 2 - f(x, y)$

Reflect the graph about xy -plane & then shift it upward 2 units.

⑤ $g(x, y) = f(x - 2, y)$

shift the graph in the direction of the positive x -axis 2 units

⑥ $g(x, y) = f(x, y + 2)$

shift the graph in the direction of the negative y -axis 2 units

⑦ $g(x, y) = f(x + 3, y - 2)$

shift the graph in the direction of the negative x -axis 3 units

and then shift in the direction of the positive y -axis 2 units

The level curves of a function f of two variables are the curves with equations $F(x,y) = k$: k : constant

The level curves $F(x,y) = k$ are just the tracer of the graph of F in the horizontal plane $z = k$ projected down to the xy -plane

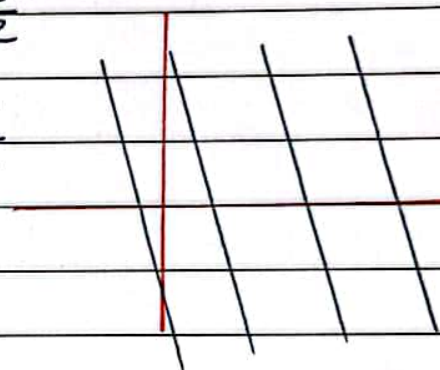
Ex: sketch the level curves of the function $F(x,y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$

$$k = -6 \Rightarrow 3x + 2y = 12 \Rightarrow \text{line with slope } -\frac{3}{2}$$

$$k = 0 \Rightarrow 3x + 2y = 6 \Rightarrow \text{line with slope } -\frac{3}{2}$$

$$k = 6 \Rightarrow 3x + 2y = 0 \Rightarrow \text{line with slope } -\frac{3}{2}$$

$$k = 12 \Rightarrow 3x + 2y = -6 \Rightarrow \text{line with slope } -\frac{3}{2}$$



Contour map

Ex: sketch the level curves of the function $F(x,y) = (9 - x^2 - y^2)^{\frac{1}{2}}$ for the values $k = 0, 1, 2, 3$

$$\sqrt{9 - x^2 - y^2} = k$$

$$k^2 = 9 - x^2 - y^2$$

$$9 - k^2 = x^2 + y^2$$

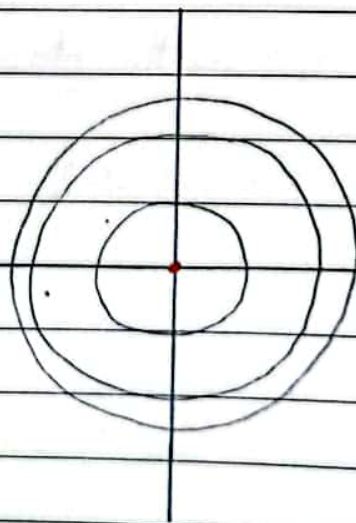
$$k = 0 \Rightarrow x^2 + y^2 = 9 \rightarrow \text{circle}$$

$$k = 1 \Rightarrow x^2 + y^2 = 8 \rightarrow \text{circle}$$

$$k = 2 \Rightarrow x^2 + y^2 = 5 \rightarrow \text{circle}$$

$$k = 3 \Rightarrow x^2 + y^2 = 0$$

$$x = 0, y = 0 \rightarrow \text{point}$$



Ex: Draw a contour map of the function $f(x,y) = 4x^2 + y^2 + 1$

$$k = 4x^2 + y^2 + 1 \Rightarrow k - 1 = 4x^2 + y^2 \Rightarrow k - 1 \geq 0 \Rightarrow k \geq 1$$

$$k = 1 \Rightarrow 4x^2 + y^2 = 0 \rightarrow \text{point}(0,0)$$

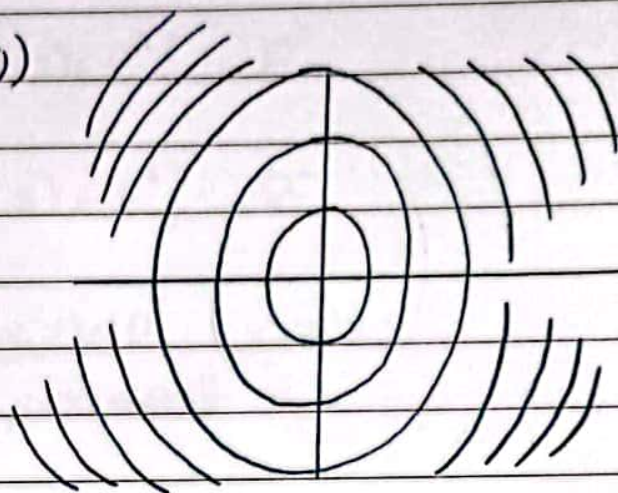
$$k = 2 \Rightarrow 4x^2 + y^2 = 1 \rightarrow \text{ellipse}$$

$$k = 3 \Rightarrow 4x^2 + y^2 = 2 \Rightarrow \text{ellipse}$$

$$2x^2 + \frac{y^2}{2} = 1$$

$$k = 4 \Rightarrow 4x^2 + y^2 = 3 \Rightarrow \text{ellipse}$$

$$2x^2 + \frac{y^2}{3} = 1$$



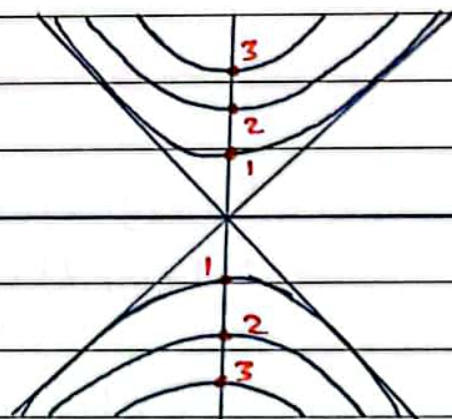
Ex: Draw a contour map of the function $f(x,y) = \sqrt{y^2 - x^2}$

$$k = \sqrt{y^2 - x^2} \Rightarrow k^2 = y^2 - x^2 \quad k \geq 0$$

$$k = 0 \Rightarrow y^2 = x^2 \Rightarrow y = x, y = -x$$

$$k > 0 \Rightarrow k^2 = y^2 - x^2 \Rightarrow \frac{y^2}{k^2} - \frac{x^2}{k^2} = 1$$

↓
hyperbolas



The surface is steep where the level curves are close together
It is somewhat flatter where they are farther apart

Functions of three or more variables

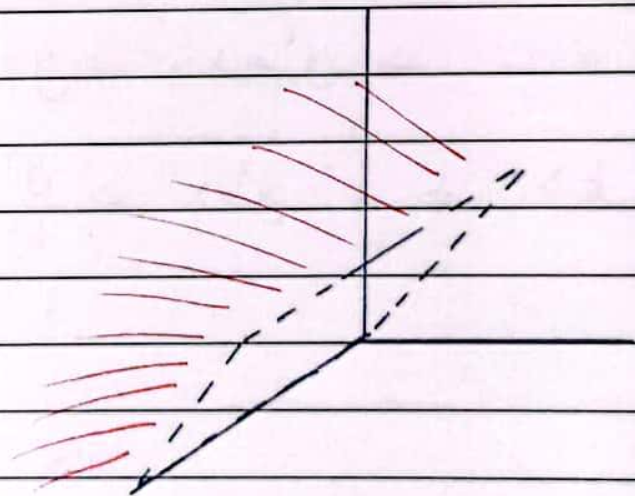
f is a rule that assigns to each ordered triple (x, y, z) in a domain in $\mathbb{R}^3$

Ex: Find & sketch the domain of the function

① $f(x, y, z) = \ln(z - y) + xy \sin z$

$$\begin{aligned}\text{Dom}(f) &= \{(x, y, z) \in \mathbb{R}^3 : z - y > 0\} \\ &= \{(x, y, z) \in \mathbb{R}^3 : z > y\}\end{aligned}$$

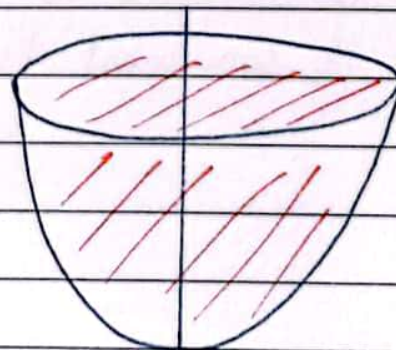
$z = y \rightarrow \text{plane}$



② $f(x, y, z) = \sqrt{z - x^2 - 2y^2}$

$$\begin{aligned}\text{Dom}(f) &= \{(x, y, z) \in \mathbb{R}^3 : z - x^2 - 2y^2 \geq 0\} \\ &= \{(x, y, z) \in \mathbb{R}^3 : z \geq x^2 + 2y^2\}\end{aligned}$$

$z = x^2 + 2y^2 \rightarrow \text{paraboloid}$



Ex: Find the Domain

① $f(x, y, z) = \frac{1}{x}$ $D(f) = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$

② $f(x, y, z) = 2e^{xyz}$ $D(f) = \{(x, y, z) \in \mathbb{R}^3\} = \mathbb{R}^3$

③ $f(x, y, z) = x + y$ $D(f) = \{(x, y, z) \in \mathbb{R}^3\} = \mathbb{R}^3$

(4) $P(x, y, z) = \sqrt{x - z}$ $D(P) = \{(x, y, z) \in \mathbb{R}^3 : x - z \geq 0\}$
 $= \{(x, y, z) \in \mathbb{R}^3 : x \geq z\}$

⑤ $f(x, y, z, w) = \sqrt{w - z}$ $D(f) = \{(x, y, z, w) \in \mathbb{R}^4 : w - z \geq 0\}$
 $= \{(x, y, z, w) \in \mathbb{R}^4 : w \geq z\}$

Ex: Find the domain & range of $F(x, y, z) = 2 + \sqrt{x^2 + 3}$

$$f(x, y, z) = 2 + \sqrt{x^2 + 3}$$

لا يوجد شيء له فائدة
لكنه فائده موجبة

$$D(F) = \{(x, y, z) \in \mathbb{R}^3\} = \mathbb{R}^3$$

$$w = 2 + \sqrt{x^2 + 3}$$

$$\textcircled{1} \sqrt{y^2+3} \geq 0 \Rightarrow 2 + \sqrt{y^2+3} \geq 2 \Rightarrow w \geq 2$$

$$\textcircled{2} \quad x^2 \geq 0 \Rightarrow x^2 + 3 \geq 3 \Rightarrow \sqrt{x^2 + 3} \geq \sqrt{3} \Rightarrow 2 + \sqrt{x^2 + 3} \geq \sqrt{3} + 2$$
$$w \geq \sqrt{3} + 2$$

$$\therefore \text{range}(f) = [2 + \sqrt{3}, \infty)$$

2

~~.....~~
 $\sqrt{3} + 2$

The level surface of a function $F(x, y, z)$ for k are the surfaces given by the eq. $F(x, y, z) = k$

Ex: Find the level surfaces for $F(x, y, z) = x^2 + y^2 + z^2$

$$x^2 + y^2 + z^2 \geq 0, \quad k \geq 0$$

$$k=0 \Rightarrow x^2 + y^2 + z^2 = 0 \rightarrow \text{point}(0, 0, 0)$$

$$k=1 \Rightarrow x^2 + y^2 + z^2 = 1 \rightarrow \text{sphere}$$

Ex: Describe the level surfaces

① $F(x, y, z) = 2x - 3y + 5z - 3$

$$k = 2x - 3y + 5z - 3$$

$$k + 3 = 2x - 3y + 5z$$

plane

② $F(x, y, z) = 2x^2 + 3y^2 + 5z^2$

$$k = 2x^2 + 3y^2 + 5z^2, \quad k \geq 0$$

$$2x^2 + 3y^2 + 5z^2 = 0$$

point $(0, 0, 0)$

③ $F(x, y, z) = y^2 + z^2$

$$k = y^2 + z^2, \quad k \geq 0$$

$$y^2 + z^2 = 0$$

$$y=0, z=0, x \in \mathbb{R}$$

surface is the x -axis / cylinder

④ $P(x, y, z) = z^2 - y^2 - x^2$

$$k = z^2 - y^2 - x^2 \quad , \quad k \in \mathbb{R}$$

$$k=0 \Rightarrow z^2 - y^2 - x^2 = 0$$

$$z^2 = x^2 + y^2$$

cone

$$k > 0 \Rightarrow k=2 \Rightarrow z^2 - x^2 - y^2 = 2$$

$$-\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = 1 \rightarrow \text{hyperboloids of two sheets}$$

$$k < 0 \Rightarrow k=-2 \Rightarrow z^2 - x^2 - y^2 = -2$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = 1 \rightarrow \text{hyperboloids of one sheet}$$

14.2) Limits & Continuity

لإثبات أن النهاية موجودة

نستخدم الشروط التالية:

- ١- التعويض المباشر
- ٢- التحليل بالاختصار
- ٣- الضرب بالمرافق
- ٤- تعويضات خاصة

لإثبات أن النهاية

غير موجودة

علينا أن نجد مسارين كليهما مارا بالنقطة P وهما من النهاية P خلال كل واحد من المسارين بحيث يكون جوابا النهاية مختلفين

let $P(a,b)$ in $\mathbb{R}^2$ and let C be a path that pass through the point $P(a,b)$ when $t = t_0$ then

$$\lim_{\substack{(x,y) \rightarrow P_0 \\ \text{along } C}} F(x,y) = \lim_{t \rightarrow t_0} F(f(t), g(t))$$

$$\lim_{\substack{(x,y) \rightarrow P_0 \\ \text{along } C_1}} F(x,y) \neq \lim_{\substack{(x,y) \rightarrow P_0 \\ \text{along } C_2}} F(x,y) \text{ then } \lim_{(x,y) \rightarrow P_0} F(x,y) \text{ does not exist}$$

Ex. Find each of the following limit

$$\textcircled{1} \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 + x^2y^2 - 6y^4}{x^2 + 3y^2} = \frac{0}{0}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + 3y^2)(x^2 - 2y^2)}{x^2 + 3y^2} = \lim_{(x,y) \rightarrow (0,0)} x^2 - 2y^2 = 0$$

$$\textcircled{2} \lim_{(x,y) \rightarrow (-1,1)} \frac{y^6 - x^2}{y^2 + x} = \frac{0}{0}$$

$$\lim_{(x,y) \rightarrow (-1,1)} \frac{(y^3 + x)(y^3 + x)}{y^2 + x} = \lim_{(x,y) \rightarrow (-1,1)} y^3 + x = -2$$

$$\textcircled{3} \lim_{(x,y) \rightarrow (4,2)} \frac{x^2 - 5xy^2 - 4y^4}{\sqrt{x} - 2y} \times \frac{\sqrt{x} + 2y}{\sqrt{x} + 2y} =$$

$$\lim_{(x,y) \rightarrow (4,2)} \frac{x^2 - 5xy^2 - 4y^4}{x - 4y^2} \times 4 = 4 \lim_{(x,y) \rightarrow (4,2)} \frac{(x - 4y^2)(x + y^2)}{x - 4y^2} = 4 \times 5 = 20$$

$$\textcircled{4} \lim_{(x,y) \rightarrow (4,2)} \frac{x^2 - 5xy^2 + 4y^4}{\sqrt{x} - 2y} = \frac{4^2 - 5(4)(4) + 4(2)^4}{\sqrt{4} - 4} = \frac{0}{0}$$

Remark :

$$\textcircled{1} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\textcircled{3} \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$$

$$\textcircled{2} \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1$$

$$\textcircled{4} \lim_{\theta \rightarrow 0} \frac{\theta}{\tan \theta} = 1$$

Ex. Find each of the following limit

① $\lim_{(x,y) \rightarrow (0,-3)} \frac{\sin(2xy^2)}{x}$

$\theta = 2xy^2$ when $(x,y) \rightarrow (0,-3)$ then $\theta \rightarrow 0$

$$\lim_{(x,y) \rightarrow (0,-3)} \frac{\sin(2xy^2)}{2xy^2} \cdot \lim_{x \rightarrow 0} \frac{2xy^2}{x} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{(x,y) \rightarrow (0,-3)} 2y^2 = 1 \cdot 18 = 18$$

② $\lim_{(x,y) \rightarrow (1,2)} \frac{4x^2 - y^2}{\tan(4x - 2y)}$

$$\lim_{(x,y) \rightarrow (1,2)} \frac{4x^2 - y^2}{4x - 2y} \cdot \lim_{(x,y) \rightarrow (1,2)} \frac{4x - 2y}{\tan(4x - 2y)} = \lim_{\theta \rightarrow 0} \frac{\theta}{\tan \theta} = 1$$

$\theta = 4x - 2y$
When $(x,y) \rightarrow (1,2)$
 $\theta \rightarrow 0$

$$\lim_{(x,y) \rightarrow (1,2)} \frac{(2x-y)(2x+y)}{2(2x-y)} \cdot \lim_{\theta \rightarrow 0} \frac{\theta}{\tan \theta} = 2$$

When $(x,y) \rightarrow (0,0)$ and we have $x^2 + y^2$ then we must have think of using the substitution $x = r \cos \theta$ & $y = r \sin \theta$ then $x^2 + y^2 = r^2$ & when $(x,y) \rightarrow (0,0)$ so $r \rightarrow 0^+$

$$\lim_{(x,y) \rightarrow (0,0)} F(x,y) = \lim_{r \rightarrow 0^+} F(r \cos \theta, r \sin \theta) \text{ where } 0 \leq \theta \leq 2\pi$$

Ex. Find the limit

③ $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{\sqrt{x^2 + y^2} + 1} - 1$ let $x = r \cos \theta, y = r \sin \theta, r \rightarrow 0^+$
 $x^2 + y^2 = r^2$

$$\lim_{r \rightarrow 0^+} \frac{r^3 \cos^2 \theta \sin \theta}{\sqrt{r^2 + 1} - 1} \cdot \frac{\sqrt{r^2 + 1} + 1}{\sqrt{r^2 + 1} + 1} = \lim_{r \rightarrow 0^+} \frac{r^3}{r^2 + 1 - 1} \cdot 2 \cos^2 \theta \sin \theta =$$

$$\lim_{r \rightarrow 0^+} 2r \cos^2 \theta \sin \theta = 0$$

$-1 \leq \cos^2 \theta \sin \theta \leq 1$
 $-r \leq r \cos^2 \theta \sin \theta \leq r$
 $0 \leq r \sin \theta \cos^2 \theta \leq 0$

} squeeze theorem
Five Apple

Ex: Find the $\lim_{(x,y) \rightarrow (1,1)} \frac{(6y-4x-1)^5 - 1}{(2x-3y)^8 - 1}$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{(6y-4x-1)^5 - 1}{(2x-3y)^8 - 1} = \lim_{(x,y) \rightarrow (1,1)} \frac{(-2(2x-3y)-1)^5 - 1}{(2x-3y)^8 - 1}$$

$\theta = 2x - 3y$ when $(x,y) \rightarrow (1,1)$ then $\theta \rightarrow -1$

$$\lim_{\theta \rightarrow -1} \frac{(-2\theta - 1)^5 - 1}{\theta^8 - 1} = \lim_{\theta \rightarrow -1} \frac{5(-2\theta - 1)^4(-2)}{8\theta^7} = \frac{5}{4}$$

Ex: ① Find $\lim_{(x,y) \rightarrow (1,-2)} \frac{x^2 + 3y + 5}{2x + y}$ along the path $C_1: y = 2x - 4$

$$\lim_{\substack{(x,y) \rightarrow (1,-2) \\ \text{along } C_1}} \frac{x^2 + 3y + 5}{2x + y} = \lim_{(x,y) \rightarrow (1,-2)} \frac{x^2 + 3(2x-4) + 5}{2x + 2x - 4} = \lim_{x \rightarrow 1} \frac{x^2 + 6x - 7}{4x - 4}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)(x+7)}{4(x-1)} = \frac{8}{4} = 2$$

② Find $\lim_{(x,y) \rightarrow (1,-2)} \frac{x^2 + 3y + 5}{2x + y}$ along the path $C_2: x = 3t, y = 1 - 9t$

$$\lim_{\substack{(x,y) \rightarrow (1,-2) \\ \text{along } C_2}} \frac{x^2 + 3y + 5}{2x + y} = \lim_{t \rightarrow \frac{1}{3}} \frac{9t^2 + 3(1-9t) + 5}{6t + 1 - 9t} = \begin{cases} 1 = 3t & -2 = 1 - 9t \\ t = \frac{1}{3} & t = \frac{1}{3} \end{cases}$$

$$\lim_{t \rightarrow \frac{1}{3}} \frac{9t^2 - 27t + 8}{1 - 3t} = \lim_{t \rightarrow \frac{1}{3}} \frac{18t - 27}{-3} = 7$$

③ Find Is $\lim_{(x,y) \rightarrow (1,-2)} \frac{x^2 + 3y + 5}{2x + y}$ exist?

$$\lim_{\substack{(x,y) \rightarrow (1,-2) \\ \text{along } C_1}} \frac{x^2 + 3y + 5}{2x + y} \neq \lim_{\substack{(x,y) \rightarrow (1,-2) \\ \text{along } C_2}} \frac{x^2 + 3y + 5}{2x + y}$$

$\therefore \lim_{(x,y) \rightarrow (1,-2)} \frac{x^2 + 3y + 5}{2x + y}$ does not exist

Ex: Find the limit $\lim_{\substack{(x,y) \rightarrow (1,0) \\ \log y = \ln x}} \frac{x \sin(e^y - 1)}{x^2 + y^2 - x}$

$$\lim_{\substack{(x,y) \rightarrow (1,0) \\ y = \ln x}} \frac{x \sin(e^y - 1)}{x^2 + y^2 - x} = \lim_{x \rightarrow 1} \frac{x \sin(e^{\ln x} - 1)}{x^2 + (\ln x)^2 - x} = \lim_{x \rightarrow 1} \frac{x \sin(x - 1)}{x^2 + (\ln x)^2 - x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{x \cos(x-1) + \sin(x-1)}{2x + \frac{2 \ln x}{x} - 1} = 1$$

Ex: Find the limit: $\lim_{(x,y) \rightarrow (2,-1)} \frac{x^2 - 2x - y^2 - 2y - 1}{(x-2)^2 + (y+1)^2}$

$$\lim_{(x,y) \rightarrow (2,-1)} \frac{x^2 - 2x - y^2 - 2y - 1}{(x-2)^2 + (y+1)^2}$$

$x = t + 2$
 $y = 0 - 1$ From the point
 $t \rightarrow 0$

$$\lim_{t \rightarrow 0} \frac{(t+2)^2 - 2(t+2) - (-1)^2 - 2(-1) - 1}{(t+2-2)^2 + (-1+1)^2}$$

$C_1: x = t + 2 \rightarrow y = -1$
 $t \rightarrow 0$

$$\lim_{t \rightarrow 0} \frac{(t+2)^2 - 2t - 4 - 1 + 2}{t^2 + 0} = \lim_{t \rightarrow 0} \frac{t^2 + 4t + 4 - 2t - 4 - 1 + 2}{t^2} = \lim_{t \rightarrow 0} \frac{t^2 + 2t}{t^2}$$

$$\lim_{t \rightarrow 0} \frac{t(t+2)}{t^2} = \lim_{t \rightarrow 0} \frac{t+2}{t} = \frac{2}{0} \text{ D.N.E}$$

Ex: Find the limit ① $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2 y^2}{x^6 - y^3 + 2z^4}$

$C_1: x = t, y = 0, z = 0, t \rightarrow 0$

$$\lim_{t \rightarrow 0} \frac{t^2 \cdot 0}{t^6} = \lim_{t \rightarrow 0} \frac{0}{t^6} = 0$$

$\text{lcm}(6, 3, 4) = 12$

لأضرب الجميع

$C_2: x = t^{\frac{12}{6}} + 0, y = t^{\frac{12}{3}} + 0, z = t^{\frac{12}{4}} + 0$
 $x = t^2, y = t^4, z = t^3, t \rightarrow 0$

$\frac{\text{lcm}}{\text{exp}} \rightarrow 5-01$ + point

$$\lim_{t \rightarrow 0} \frac{t^4 \cdot t^8}{t^{12} - t^{12} + 2t^{12}} = \lim_{t \rightarrow 0} \frac{t^{12}}{2t^{12}} = \frac{1}{2}$$

$$\therefore \lim_{\substack{(x,y,z) \rightarrow (0,0,0) \\ \text{along } C_1}} \frac{x^2 y^2}{x^6 - y^3 + 2z^4} \neq \lim_{\substack{(x,y,z) \rightarrow (0,0,0) \\ \text{along } C_2}} \frac{x^2 y^2}{x^6 - y^3 + 2z^4} \Rightarrow \text{D.N.E}$$

$$(2) \lim_{(x,y,z) \rightarrow (1,-2,3)} \frac{(x-1)^2 (y+2)(z-3)^2}{(x-1)^3 + 3(y+2)^5 + (z-3)^{15}}$$

$$C_1: x = t+1, y = -2, z = 3 / t \rightarrow 0$$

$$\lim_{\substack{(x,y,z) \rightarrow (1,-2,3) \\ \text{along } C_1}} \frac{(x-1)^2 (y+2)(z-3)^2}{(x-1)^3 + 3(y+2)^5 + (z-3)^{15}} = \lim_{t \rightarrow 0} \frac{t^2 \cdot 0 \cdot 0}{t^3 + 0} = \frac{0}{t^3} = 0$$

$$\text{lcm}(3, 5, 15) = 15 / x = t^{\frac{15}{3}} + 1, y = t^{\frac{15}{5}} - 2, z = t^{\frac{15}{15}} + 3 / t \rightarrow 0$$

$$C_2: x = t^5 + 1, y = t^3 - 2, z = t + 3$$

$$\lim_{\substack{(x,y,z) \rightarrow (1,-2,3) \\ \text{along } C_2}} \frac{(x-1)^2 (y+2)(z-3)^2}{(x-1)^3 + 3(y+2)^5 + (z-3)^{15}} = \lim_{t \rightarrow 0} \frac{t^{10} \cdot t^3 \cdot t^2}{t^{15} + 3t^{15} + t^{15}} = \lim_{t \rightarrow 0} \frac{t^{15}}{5t^{15}} = \frac{1}{5}$$

$$\therefore \lim_{(x,y,z) \rightarrow (1,-2,3)} \frac{(x-1)^2 (y+2)(z-3)^2}{(x-1)^3 + 3(y+2)^5 + (z-3)^{15}} \text{ D.N.E}$$

A function is said to be continuous at a point (a,b) in $\text{Dom}(f)$ if $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$

A function is said to be continuous on a set $S \subseteq \text{Dom}(f)$ if $f(x,y)$ is continuous at every point in S

continuous on $\mathbb{R}^2 \Rightarrow$ continuous everywhere

continuous on $\mathbb{R}^3 \Rightarrow$ continuous everywhere

Ex:

$$\textcircled{1} f(x,y) = \frac{x^4 + x^2 y - 5}{y^2 + 1}$$

$\text{Dom}(f) = \mathbb{R}^2 \Rightarrow f$ is continuous everywhere

$$\textcircled{2} f(x,y) = \frac{ye^x - 5}{x^2 + y^2}$$

$\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0\} \Rightarrow f$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\textcircled{3} f(x,y,z) = \frac{z \ln y - 5x}{x - 2y - z}$$

$\text{Dom}(f) = \{(x,y,z) \in \mathbb{R}^3 : x - 2y - z \neq 0, y > 0\}$

$$\textcircled{4} f(x,y,z) = 2$$

$\text{Dom}(f) = \mathbb{R}^3 \Rightarrow f$ is continuous everywhere

Ex: Find the region on which the function f is continuous

$$f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$C_1: x=t, y=0 / t \rightarrow 0$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } C_1}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{t \rightarrow 0} \frac{t^2 - 0}{t^2 + 0} = 1$$

$$C_2: x=0, y=t / t \rightarrow 0$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } C_2}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{t \rightarrow 0} \frac{0 - t^2}{0 + t^2} = -1$$

$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist

$f(x,y)$ D.N.C on $(0,0)$

$f(x,y)$ is cont. on $\mathbb{R}^2 \setminus \{(0,0)\}$

Ex: Find the value of k that the function is cont on $(0,0)$

$$f(x,y) = \begin{cases} \frac{1 - \cos \sqrt{x^2 + y^2}}{x^2 + y^2} & , (x,y) \neq (0,0) \\ k & , (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{1 - \cos \sqrt{x^2 + y^2}}{x^2 + y^2} = k$$

c: $x = t, y = 0$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } c}} \frac{1 - \cos \sqrt{x^2 + y^2}}{x^2 + y^2} = \lim_{t \rightarrow 0} \frac{1 - \cos \sqrt{t^2}}{t^2} = \lim_{t \rightarrow 0} \frac{1 - \cos t}{t^2}$$

$$\lim_{t \rightarrow 0} \frac{\sin t}{2t} = \lim_{t \rightarrow 0} \frac{\cos t}{2} = \frac{1}{2} \Rightarrow k = \frac{1}{2}$$

Ex: Find the value of k that the function f is cont. everywhere

$$f(x,y) = \begin{cases} \frac{kx^2 - 2y^2}{x^2 + y^2} & , (x,y) \neq (0,0) \\ -2 & , (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{kx^2 - 2y^2}{x^2 + y^2} = -2$$

c: $x = t, y = 0$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } c}} \frac{kx^2 - 2y^2}{x^2 + y^2} = -2 \Rightarrow \lim_{t \rightarrow 0} \frac{t^2 k - 0}{t^2 + 0} = -2$$

$$\lim_{t \rightarrow 0} \frac{t^2 k}{t^2} = -2 \Rightarrow k = -2$$

Ex: Find the value of a such that the function is cont at $(0,2)$

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{x} & , x \neq 0 \\ a & , x = 0 \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,2)} \frac{\sin(xy)}{x} = a$$

$$C: x = t, y = 2$$

$$\lim_{\substack{(x,y) \rightarrow (0,2) \\ \text{along } C}} \frac{\sin(xy)}{x} = \lim_{t \rightarrow 0} \frac{\sin 2t}{t} = 2 \Rightarrow a = 2$$

Ex: Find the value of k such that f is cont at $(1,1)$

$$f(x,y) = \begin{cases} \frac{\sqrt{xy+8} - 3}{xy-1} & , xy \neq 1 \\ k & , xy = 1 \end{cases}$$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{\sqrt{xy+8} - 3}{xy-1} = k$$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{\sqrt{xy+8} - 3}{xy-1} \cdot \frac{\sqrt{xy+8} + 3}{\sqrt{xy+8} + 3} = \lim_{(x,y) \rightarrow (1,1)} \frac{\cancel{xy+8} - 9}{\cancel{xy-1}} \cdot \frac{1}{6}$$

$$= \frac{1}{6} \Rightarrow k = \frac{1}{6}$$

14.3) Partial Derivative

The partial derivative of F :

① with respect to x at point (a,b) written as $F_x(a,b)$

$$F_x(a,b) = \lim_{x \rightarrow a} \frac{F(x,b) - F(a,b)}{x - a}$$

② with respect to y at point (a,b) written as $F_y(a,b)$

$$F_y(a,b) = \lim_{y \rightarrow b} \frac{F(a,y) - F(a,b)}{y - b}$$

$F_x(a,b) = g'(a)$ where $g(x) = F(x,b)$

$F_y(a,b) = h'(b)$ where $h(y) = F(a,y)$

Ex: Find $F_x(1,0)$ & $F_y(1,0)$ where $F(x,y) = \sqrt{x^4 + y^2 + 3}$

$$F_x(a,b) = F_x(1,0) = g'(1) \text{ where } g(x) = F(x,0) = \sqrt{x^4 + 0 + 3} = \sqrt{x^4 + 3}$$

$$g'(x) = \frac{4x^3}{2\sqrt{x^4 + 3}} \Rightarrow g'(1) = \frac{4}{4} = 1$$

$$F_y(1,0) = h'(0) \Rightarrow h(y) = F(1,y) = \sqrt{1 + y^2 + 3} = \sqrt{y^2 + 4}$$

$$h'(y) = \frac{3y^2}{2\sqrt{y^2 + 4}} \Rightarrow h'(0) = 0$$

Ex. Find $P_x(0,0)$ where $f(x,y) = 3x + \sqrt[3]{8x^3 + 27y^6}$

$$g(x) = 3x + 2x = 5x$$

$$g'(x) = 5$$

Ex. Find $P_x(0,0)$ where $f(x,y) = \sqrt{x^2 + y^2}$ if it exist

$$g(x) = \sqrt{x^2} = |x|$$

since $|x|$ is not differentiable at $x=0$ then $P_x(0,0)$ D.N.E
∴ the partial derivative of f with respect to x D.N.E at $(0,0)$

Ex. Find $P_y(0,0)$ where

$$f(x,y) = \begin{cases} \frac{3x^2 + xy + y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$h(y) = \frac{y^3}{y^2} = y$$

$$h'(y) = 1 \Rightarrow h'(0) = 1$$

by definition:

$$P_y(0,0) = \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y \rightarrow 0} \frac{\frac{y^3}{y^2}}{y} = 1$$

Ex. Find $P_y(0,0)$ where $f(x,y) = \begin{cases} \frac{3x^2 + xy + y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

$$h(y) = \frac{y^3}{y^2} = y$$

$$h'(y) = 1$$

$f(0,y)$ is dis cont at $(x,y) = 0$

$P_y(0,0)$ does not exist

(المعادلة لا تعبر عن دالة واحدة)

Ex: Find $P_x(x,y)$ & $P_y(x,y)$ where $P(x,y) = xy^4e^{3x} + \cos 2y$

∂/∂y

$$P_x(x,y) = xy^4 \cdot 3e^{3x} + e^{3x} (\cancel{xy^4}) + 0 = 3xy^4e^{3x} + y^4e^{3x}$$

∂/∂x

$$P_y(x,y) = xe^{3x}(4y^3) + (-\sin(2y)(2)) = 4xy^3e^{3x} - 2\sin(2y)$$

Ex: Find $P_x(1,0)$ & $P_y(1,-1)$ where

$$P(x,y) = \begin{cases} \frac{3x^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$P_x(x,y) = \frac{(x^2+y^2) \cdot 9x^2 - 3x^3(2x)}{(x^2+y^2)^2} = \frac{3x^4 + 9x^2y^2}{(x^2+y^2)^2} \Rightarrow P_x(1,0) = 3$$

$$P_y(x,y) = \frac{(x^2+y^2)(0) - (3x^3 \cdot 2y)}{(x^2+y^2)^2} = \frac{-6x^3y}{(x^2+y^2)^2} \Rightarrow P_y(1,-1) = \frac{6}{4} = \frac{3}{2}$$

Ex: Find $P_x(x,y)$ where

$$P(x,y) = \begin{cases} \frac{3x^3 + xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

(IF $P(x,y) \neq (0,0)$)

$$P_x(x,y) = \frac{(x^2+y^2)(9x^2+y) - (3x^3+xy)(2x)}{(x^2+y^2)^2} = \frac{3x^4 + 9x^2y^2 - 2x^2y}{(x^2+y^2)^2}$$

(IF $(x,y) = (0,0)$)

$$g(x) = \frac{3x^3}{y^2} = 3x$$

$$g'(x) = 3 \rightarrow g'(0) = 3$$

Ex: Find $\lim_{h \rightarrow 0} \frac{P(1+h, -1) - P(1, -1)}{h}$ where

$$P(x, y) = \sqrt[5]{x^7 - y^2 + 1}$$

$$P_x(1, -1)$$

$$P_x = \frac{1}{5} (x^7 - y^2 + 1)^{-\frac{4}{5}} \cdot 7x^6$$

$$P_x(1, -1) = \frac{1}{5} (1 - 1 + 1)^{-\frac{4}{5}} \cdot 7 = \frac{7}{5}$$

إذا كانت h موجودة في المقام

الأول $P_x \leftarrow$

إذا كانت موجودة في المقام

الثاني $P_y \leftarrow$

Ex: let $P(x, y)$ be a function such that $P_x = 2xy$, $P_y = x^2 + 2y$ and $P(1, 1) = 8$ find $(0, 2)$

$$P_x = 2xy \Rightarrow \overset{\text{integrate}}{P(x, y)} = \int 2xy \, dx + g(y)$$

$$P(x, y) = x^2 y + g(y)$$

$$\overset{\text{differentiate}}{P_y} = x^2 + g'(y) \Rightarrow x^2 + 2y = x^2 + g'(y)$$
$$2y = g'(y)$$

$$g(y) = \int 2y \, dy \Rightarrow g(y) = y^2 + C$$

$$P(1, 1) = 8 \Rightarrow 1(1) + 1^2 + C = 8$$
$$2 + C = 8$$

$$C = 6$$

$$P(x, y) = x^2 y + y^2 + 6$$

$$f(0, 2) = 0 + 4 + 6 = 10$$

Ex: Find $P_z(-1, 1, e^2)$, where $P(x, y, z) = e^{2xy} \ln z$

$$P_z = \frac{e^{2xy}}{z} \Rightarrow P_z(-1, 1, e^2) = \frac{e^{2(-1)(1)}}{e^2} = e^{-4}$$

There are several forms of partial derivatives of $z = P(x, y)$

$$P_x(x, y) = z_x = \frac{\partial}{\partial x} P(x, y) = \frac{\partial P}{\partial x} = \frac{\partial z}{\partial x} = D_x P = D_1 P$$

$$P_y(x, y) = P_y = z_y = \frac{\partial}{\partial y} P(x, y) = \frac{\partial P}{\partial y} = \frac{\partial z}{\partial y} = D_y P = D_2 P$$

Ex: Find $\frac{\partial P}{\partial x}$ & $\frac{\partial P}{\partial y}$ at the point $(\pi, \sqrt{3})$ where

$$P(x, y) = \sin\left(\frac{x}{y^2+1}\right)$$

$$\frac{\partial P}{\partial x} = \cos\left(\frac{x}{y^2+1}\right) \cdot \left(\frac{1}{y^2+1}\right) \Rightarrow \frac{\partial P}{\partial x} \Big|_{(\pi, \sqrt{3})} = \frac{1}{4\sqrt{2}}$$

$$\frac{\partial P}{\partial y} = \cos\left(\frac{x}{y^2+1}\right) \cdot \left(\frac{y^2+1 - x \cdot 2y}{(y^2+1)^2}\right) \Rightarrow \frac{\partial P}{\partial y} \Big|_{(\pi, \sqrt{3})} = -\frac{\pi\sqrt{3}}{8\sqrt{2}}$$

If $P(x, y) = 4 - x^2 - 2y^2$, find $P_x(1, 1)$ & $P_y(1, 1)$ & interpret these numbers as slopes

$$P_x = -2x \Rightarrow P_x(1, 1) = -2 \text{ \& } P_y = -4y \Rightarrow P_y(1, 1) = -4$$

Determine the signs of the partial derivatives for the function whose graph is shown in the figure

① $P_x(1, 2) \rightarrow$ positive

③ $P_x(-1, 2) \rightarrow$ negative

② $P_y(1, 2) \rightarrow$ negative

④ $P_y(-1, 2) \rightarrow$ negative

Ex: Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ if z is a function in x & y which

is defined by $\frac{x^3 - y^4 + z^3}{1 - 6xyz} = 1$

$$\frac{x^3 - y^4 + z^3}{1 - 6xyz} = 1 \Rightarrow x^3 - y^4 + z^3 = 1 - 6xyz \Rightarrow x^3 - y^4 + z^3 + 6xyz = 1$$

$$\textcircled{1} \frac{\partial z}{\partial x} \Rightarrow 3x^2 + 3z^2 \cdot z_x + 6xy z_x + 6yz = 0$$

$$z_x(3z^2 + 6xy) = -(3x^2 + 6yz) \Rightarrow z_x = -\frac{3x^2 + 6yz}{3z^2 + 6xy}$$

$$\textcircled{2} \frac{\partial z}{\partial y} \Rightarrow -4y^3 + 3z^2 \cdot z_y + 6xy z_y + 6xz = 0$$

$$z_y(3z^2 + 6xy) = 4y^3 - 6xz \Rightarrow z_y = \frac{4y^3 - 6xz}{3z^2 + 6xy}$$

Higher Derivative 20: Let z

Let $z = f(x, y)$ Then the second partial derivatives of f are:

$$z_{xx} = f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} \quad / \quad z_{yy} = f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2}$$

$$z_{xy} = f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} \quad / \quad z_{yx} = f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

Ex: Find the second partial derivative of $f(x, y) = x^3 + x^2y^3 - 2y^2$

$$f_x = 3x^2 + 2xy^3 \quad \& \quad f_y = 3y^2x^2 - 4y$$

$$f_{xy} = (f_x)_y = (3x^2 + 2xy^3)_y = 6xy^2$$

$$f_{yx} = (f_y)_x = (3y^2x^2 - 4y)_x = 6xy^2$$

Ex. Find $\frac{\partial^2 f}{\partial y \partial x}$ & $\frac{\partial^2 f}{\partial x \partial y}$ at the origin where

$$f(x, y) = 2x^3 \sqrt[3]{x^3 - 27y^3}$$

$$f_x = 2x \cdot \frac{1}{3} (x^3 - 27y^3)^{-\frac{2}{3}} \cdot 3x^2 + 2 \sqrt[3]{x^3 - 27y^3}$$

$$f_x = \frac{2x^3}{(x^3 - 27y^3)^{\frac{2}{3}}} + 2 \sqrt[3]{x^3 - 27y^3}$$

$$f_x(0, y) = 0 + -6y = -6y \Rightarrow f_{xy}(0, y) = -6 \Rightarrow f_{xy}(0, 0) = -6$$

$$f_y = 2x \cdot \frac{1}{3} (x^3 - 27y^3)^{-\frac{2}{3}} \cdot (-27(3)y^2) = \frac{-54xy^2}{(x^3 - 27y^3)^{\frac{2}{3}}}$$

$$f_y(x, 0) = \frac{0}{(x^3)^{\frac{2}{3}}} = 0 \Rightarrow f_{yx}(0, 0) = 0$$

Ex. I $f(x, y, z) = \frac{y^2 e^{xyz}}{8x^2}$ Find $f_{yxxzx}(1, 2, 0)$

$$f_z = \frac{y^2 (3xy) e^{xyz}}{8x^2}$$

$$f_z(x, y, 0) = \frac{y^2 (3xy) e^{3xy(0)}}{8x^2} = \frac{3xy^3}{8x^2} = \frac{3y^3}{8x}$$

$$f_{zy}(x, y, 0) = \frac{9y^2}{8x} \Rightarrow f(x, 2, 0) = \frac{9 \times 4}{8x} = \frac{9}{2x} \Rightarrow$$

$$f_{zy}(x, 2, 0) = \frac{9}{2} x^{-1} \Rightarrow f_{zyx}(x, 2, 0) = -\frac{9}{2} x^{-2}$$

$$f_{zxx}(x, 2, 0) = 9x^{-3} \Rightarrow f_{zyxxx}(x, 2, 0) = -27x^{-4}$$

$$f_{yxxzx}(1, 2, 0) = -27$$

$$f_{xxxyyyyy}(e,2)$$

Ex: IF $f(x,y) = x^3 y^5 - \frac{xy^2}{x+\ln x}$ Find $\left(\frac{\partial^6 f}{\partial y^4 \partial x^2} \right)_{(e,2)}$

$$f_{xxxyyyy}(e,2) = f_{yyyyxx}(e,2)$$

$$f_y = 5x^3 y^4 - \frac{2xy}{x+\ln x} \Rightarrow f_{yy} = 20x^3 y^3 - \frac{2x}{x+\ln x} \Rightarrow$$

$$f_{yyy} = 60x^3 y^2 \Rightarrow f_{yyyyxx} = 120x^3 y \Rightarrow f_{yyyy}(x,2) = 240x^3$$

$$\Rightarrow f_{yyyyx}(x,2) = 720x^2 \Rightarrow f_{yyyyxx}(x,2) = 1440x \Big|_{(e,2)} = 1440e$$

Ex: Find $\frac{\partial^{102}}{\partial y^{63} \partial x^{40}} (x^{10} \sin xy + x^{50})$ at the point $(-1,0)$

$$f(x,y) = x^{10} \sin xy + x^{50}$$

$$f_y = x^{10} (x \cos xy)$$

$$f_{y \dots y}^{60 \text{ times}} = x^{10} (x^{60} \sin xy)$$

$$f_{yy} = x^{10} (-x^2 \sin xy) \Rightarrow$$

$$f_{y \dots y}^{61 \text{ times}} = x^{10} (x^{61} \cos xy)$$

$$f_{yyy} = x^{10} (-x^3 \cos xy)$$

$$f_{y \dots y}^{62} = x^{10} (-x^{62} \sin xy)$$

$$f_{yyyy} = x^{10} (x^4 \sin xy)$$

$$f_{y \dots y}^{63} = x^{10} (-x^{63} \cos xy)$$

$$f_{y \dots y}^{63}(x,0) = x^{10} (-x^{63} \cos(x \cdot 0)) = -x^{73}$$

$$f_{y \dots y}^{63} x^{40} x(x,0) = -73(72)(71) \dots (73-39) x^{73-40}$$

$$f_{y \dots y}^{63} x^{40} x(x,0) = -73(72)(71) \dots (34) x^{33} = \frac{-73!}{33!} x^{33}$$

$$\frac{d}{dx} \int_{g(x)}^{h(x)} F(t) dt = F(h(x)) h'(x) - F(g(x)) g'(x)$$

Ex. $P(x,y) = \int_y^{xy} \cos e^t dt$ Find $P_{xy}(0,0)$

$$P_x = \cos(e^{xy}) \frac{\partial}{\partial x}(xy) - \cos(e^y) \frac{\partial}{\partial x}(y) = y \cos e^{xy} - \cos e^y(0)$$

$$P_x = y \cos e^{xy} \Rightarrow P_x(0,y) = y \cos(e^{0y}) = y \cos 1$$

$$P_{xy} = \cos 1 \Rightarrow P_{xy}(0,0) = \cos 1$$

The Laplace eq is the partial differential eq

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{where } u(x,y) \text{ is function in } x \text{ \& } y$$

The wave eq is the partial differential eq

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Ex: ① Show that $u(x,y) = e^x \sin y$ is a solution to the Laplace eq

$$u(x,y) = e^x \sin y \Rightarrow u_x = e^x \sin y \Rightarrow u_{xx} = e^x \sin y$$

$$u(x,y) = e^x \sin y \Rightarrow u_y = e^x \cos y \Rightarrow u_{yy} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0 \quad \checkmark$$

② show that $u(x,t) = \sin(x-at)$ is a solution to the wave eq.

$$u(x,t) = \sin(x-at) \Rightarrow u_t = -a \cos(x-at) \Rightarrow u_{tt} = -a^2 \sin(x-at)$$

$$u(x,t) = \sin(x-at) \Rightarrow u_x = \cos(x-at) \Rightarrow u_{xx} = -\sin(x-at)$$

$$\frac{\partial^2 u}{\partial t^2} = -a^2 \sin(x-at)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = -a^2 \sin(x-at)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = -a^2 \sin(x-at) \quad \checkmark$$

14.5) The Chain Rule

قاعدة السلسلة

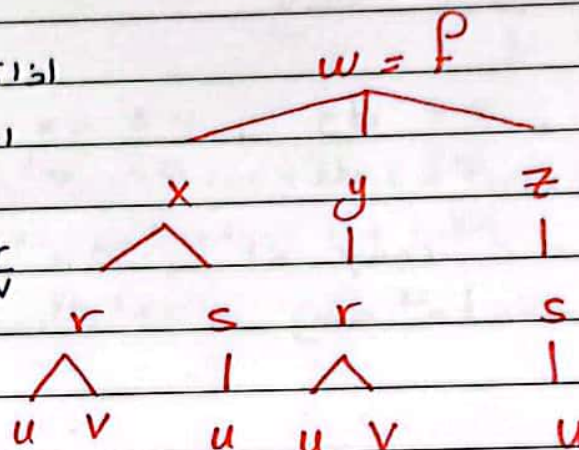
اقترب بالاضافة u, v اقترب بالاضافة s r y x, s

let $w = P(x, y, z)$, $x = x(r, s)$, $y = y(r)$, $z = z(s)$, $r = r(u, v)$, $s = s(u)$

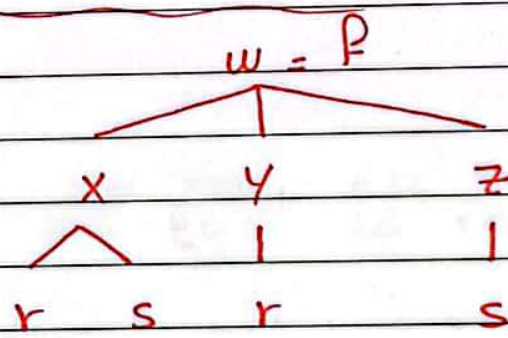
$$\frac{\partial w}{\partial u} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial r} \frac{\partial r}{\partial u} + \frac{\partial P}{\partial x} \frac{\partial x}{\partial s} \frac{\partial s}{\partial u} + \frac{\partial P}{\partial y} \frac{dy}{dr} \frac{\partial r}{\partial u} + \frac{\partial P}{\partial z} \frac{dz}{ds} \frac{\partial s}{\partial u}$$

اذا كانه الاقترب بتغير واحد لا يكونه اشتقاقه
اشتقاقه جزئي

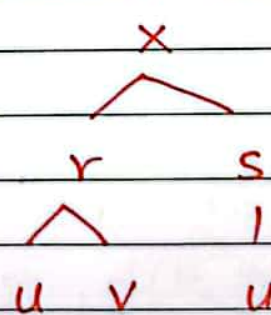
$$\frac{\partial w}{\partial v} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial r} \frac{\partial r}{\partial v} + \frac{\partial P}{\partial y} \frac{dy}{dr} \frac{\partial r}{\partial v}$$



$$\frac{\partial w}{\partial s} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial P}{\partial z} \frac{dz}{ds}$$



$$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial r} \frac{\partial r}{\partial u} + \frac{\partial x}{\partial s} \frac{\partial s}{\partial u}$$



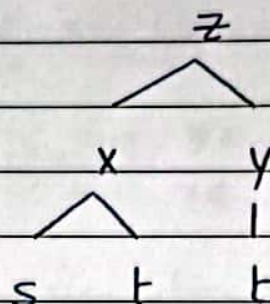
$$\frac{\partial x}{\partial v} = \frac{\partial x}{\partial r} \frac{\partial r}{\partial v}$$

Ex. $z = e^{2x} \sin y$, $x = st^2$, $y = t^3$ find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$

$$\textcircled{1} \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s}$$

$$= 2e^{2x} \sin y \cdot t^2$$

$$= 2t^2 e^{2x} \sin y$$



$$\textcircled{2} \frac{\partial z}{\partial t} = \frac{\partial z}{\partial y} \frac{dy}{dt} + \frac{\partial z}{\partial x} \frac{\partial x}{\partial t}$$

$$= e^{2x} \cos y \cdot 3t^2 + 2e^{2x} \sin y \cdot 2st$$

$$= 4ste^{2x} \sin y + 3t^2 e^{2x} \cos y = 4stz + 3t^2 e^{2x} \cos y$$

Ex. $u = x^4 y + y^2 z^3$, $x = se^{2t}$, $y = r^2 se^{-t}$, $z = r \cos t$

Find $\frac{\partial u}{\partial s}$ & $\frac{\partial u}{\partial t}$ when $r=2$, $s=1$, $t=0$ $x=1$ / $y=4$

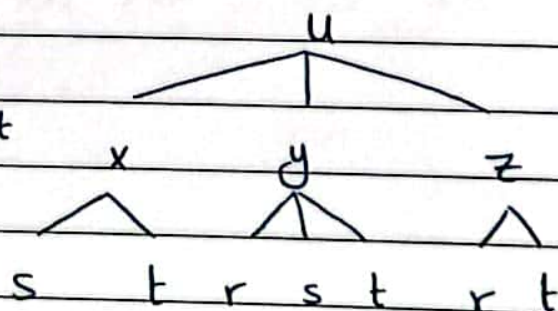
$$z=2$$

$$\textcircled{1} \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}$$

$$= 4x^3 y \cdot e^{2t} + (x^4 + 2yz^3) \cdot r e^{-t}$$

$$= 16 + 260$$

$$= 276$$



$$\textcircled{2} \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}$$

$$= (4x^3 y \cdot 2se^{2t}) + (x^4 + 2yz^3) \cdot (-r^2 se^{-t}) + (3z^2 y^2) \cdot (-rsint)$$

$$= 32 + 65 \cdot -4 + 0$$

$$= -228$$

Ex: $w = \ln \sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$ Find $\frac{dw}{dt}$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \end{aligned}$$

$= \frac{1}{2} \left(\frac{2x}{x^2 + y^2 + z^2} \right) \cos t + \frac{-1}{2} \left(\frac{2y}{x^2 + y^2 + z^2} \right) \sin t +$

$\begin{array}{ccc} & w & \\ & \swarrow \quad \searrow & \\ x & & y & z \\ | & & | & | \\ t & & t & t \end{array}$

$$\frac{1}{2} \left(\frac{2z}{x^2 + y^2 + z^2} \right) \sec^2 t =$$

$$\frac{x \cos t - y \sin t + z \sec^2 t}{x^2 + y^2 + z^2} = \frac{\sin t \cos t - \cos t \sin t + \tan t \sec^2 t}{\sin^2 t + \cos^2 t + \tan^2 t} =$$

$$\frac{\tan t \sec^2 t}{1 + \tan^2 t} = \frac{\tan t \sec^2 t}{\sec^2 t} = \tan t$$

Ex: $z = f(x, y)$, $x = g(t)$, $y = h(t)$, $z = g(3) = 2$ / $h(3) = 7$
 $g'(3) = 5$ / $h'(3) = -4$
 $f_x(2, 7) = 6$ & $f_y(2, 7) = -8$ Find $\frac{dz}{dt}$ when $t = 3$

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= f_x(x, y) g'(t) + f_y(x, y) h'(t) \\ &= f_x(2, 7) g'(3) + f_y(2, 7) h'(3) \\ &= 62 \end{aligned}$$

$\begin{array}{ccc} & z = f & \\ & \swarrow \quad \searrow & \\ x & & y \\ | & & | \\ t & & t \end{array}$

$\begin{cases} x = 2 \\ y = 7 \end{cases}$

$W(s, t) = F(u(s, t), v(s, t))$, $F_u(2, 3) = -1$, $F_v(2, 3) = 10$
 $u(1, 0) = 2$, $v(1, 0) = 3$, $u_s(1, 0) = -2$, $v_s(1, 0) = 5$, $u_t(1, 0) = 6$, $v_t(1, 0) = 4$
 Find $W_t(1, 0)$ & $W_s(1, 0)$

$$\begin{aligned} W_t &= \frac{\partial F}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial t} \\ &= -1 \times 6 + 10 \times 4 = 34 \end{aligned}$$

$\begin{array}{ccc} & u & \\ & \swarrow \quad \searrow & \\ s & & t \end{array}$

$\begin{array}{ccc} & v & \\ & \swarrow \quad \searrow & \\ s & & t \end{array}$

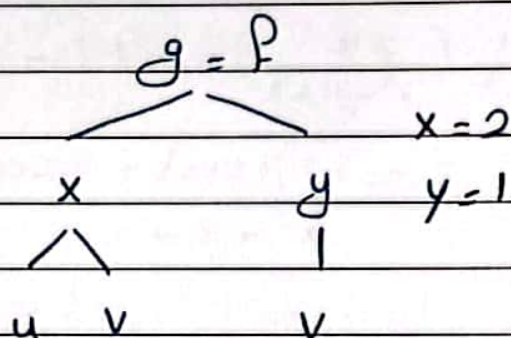
$$W_s = \frac{\partial F}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial s} = -1 \times -2 + 10 \times 5 = 52$$

~~Ex: $x = e^u + \cos v$, $y = 1 + \sin v$~~

Ex: Suppose that $P(x, y)$ is differentiable find $g_u(0,0)$ & $g_v(0,0)$ where $g(u, v) = P(e^u + \cos v, 1 + \sin v)$

let $x = e^u + \cos v$, $y = 1 + \sin v$ $u=0/v=0$ $g(u, v) = P(x, y)$

$$g_u = P_x x_u = P_x(2, 1) e^u = 2e^0 = 2$$



$$g_v = P_x x_v + P_y y_v$$

$$= P_x(2, 1) \sin v + P_y(1, 2) \cos v$$

$$= -2 \times 0 + 7 \times 1$$

$$= 7$$

	P	g	P_x	P_y
$(0, 0)$	3	6	5	8
$(2, 1)$	6	3	2	7

Ex: let $z = P(x-y)$ show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

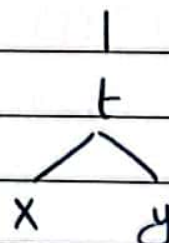
$t = x - y \rightarrow z = P(t)$

$z = P(t)$

$$\frac{\partial z}{\partial x} = \frac{\partial P}{\partial t} \frac{dt}{dx} = P'(t) \cdot 1 = P'(t)$$

$$\frac{\partial z}{\partial y} = \frac{\partial P}{\partial t} \frac{dt}{dy} = P'(t) \cdot -1 = -P'(t)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1 - 1 = 0 \quad \checkmark \quad \#$$

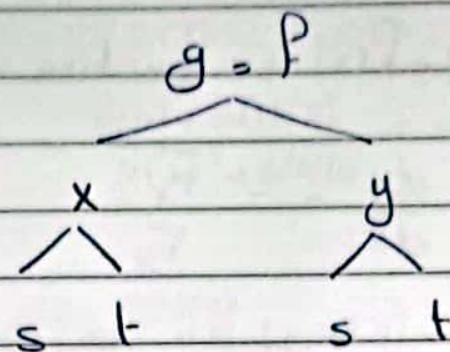


Ex: let $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ show that $t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0$

$$x = s^2 - t^2, \quad y = t^2 - s^2 \rightarrow g(s, t) = f(x, y)$$

$$\begin{aligned} \frac{\partial g}{\partial s} &= f_x x_s + f_y y_s \\ &= 2s f_x + -2s f_y \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial t} &= f_x x_t + f_y y_t \\ &= -2t f_x + 2t f_y \end{aligned}$$



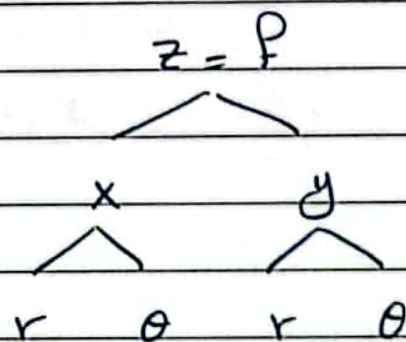
$$\begin{aligned} t(2s f_x - 2s f_y) + s(-2t f_x + 2t f_y) &= \\ 2ts f_x - 2ts f_y + -2ts f_x + 2st f_y &= 0 \quad \checkmark \quad \# \end{aligned}$$

Ex: let $z = f(x, y)$ be with continuous second order partial derivatives such that $x = r \cos \theta$, $y = r \sin \theta$ show that

$$\textcircled{1} \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 = \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2$$

$$\frac{\partial z}{\partial r} = f_x x_r + f_y y_r = \cos \theta f_x + \sin \theta f_y$$

$$\frac{\partial z}{\partial \theta} = f_x x_\theta + f_y y_\theta = -r \sin \theta f_x + r \cos \theta f_y$$



$$\left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 = (\cos \theta f_x + \sin \theta f_y)^2 + \frac{1}{r^2} (-r(\sin \theta f_x - \cos \theta f_y))^2$$

$$= (f_x)^2 + (f_y)^2 = \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2$$

Implicit Differentiation.

⇒ Implicit Function

Let $y = f(x)$ is a function defined implicitly by the relation $F(x, y) = 0$

$$\frac{dy}{dx} = - \frac{F_x}{F_y}$$

Let $z = F(x, y)$ is a function defined implicitly by the relation $F(x, y, z) = 0$

$$\frac{\partial z}{\partial x} = - \frac{F_x}{F_z} \quad \& \quad \frac{\partial z}{\partial y} = - \frac{F_y}{F_z}$$

Ex: Find y' at $x=0$ if $\frac{x^3+y^3+1}{6y} = x$

① $x=0 \rightarrow y=-1$

② simplify $\Rightarrow x^3+y^3+1 = 6xy \rightarrow x^3+y^3-6xy+1=0$

③ $F(x, y) = x^3+y^3-6xy+1$

$$y' = - \frac{F_x}{F_y} = - \frac{3x^2 - 6y}{3y^2 - 6x} \Rightarrow y' \Big|_{\substack{x=0 \\ y=-1}} = -2$$

Ex. IF $x^3+y^3+z^3+6xyz=1$ find

① $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = - \frac{F_x}{F_z} = - \frac{3x^2+6yz}{3z^2+6xy} \quad \& \quad \frac{\partial z}{\partial y} = - \frac{F_y}{F_z} = - \frac{3y^2+6xz}{3z^2+6xy}$$

② $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at the point (0,1)

$$x^3 + y^3 + z^3 + 6xyz - 1 = 0 \Rightarrow 0^3 + 1^3 + z^3 + 6(0)(1)z - 1 = 0$$
$$z^3 = 8 \rightarrow z = 2$$

$$\frac{\partial z}{\partial x} = - \frac{3x^2 + 6yz}{3z^2 + 6xy} \Rightarrow \frac{\partial z}{\partial x} \Big|_{(0,1,2)} = - \frac{3(0) + 6(1)(2)}{3(2)^2 + 6(0)(1)} = -1$$

$$\frac{\partial z}{\partial y} = - \frac{3y^2 + 6xz}{3z^2 + 6xy} \Rightarrow \frac{\partial z}{\partial y} \Big|_{(0,1,2)} = - \frac{3(1) + 6(0)(2)}{3(2)^2 + 6(0)(1)} = -\frac{1}{4}$$

Ex: Suppose that the equation $F(x,y,z)=0$ implicitly defines each of the three variables x, y, z as a function of the other two. If F_x, F_y, F_z are nonzero show that $\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1$

$$\frac{\partial z}{\partial x} = - \frac{F_x}{F_z}, \quad \frac{\partial x}{\partial y} = - \frac{F_y}{F_x}, \quad \frac{\partial y}{\partial z} = - \frac{F_z}{F_y}$$

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = \left(- \frac{F_x}{F_z}\right) \left(- \frac{F_y}{F_x}\right) \left(- \frac{F_z}{F_y}\right) = -1$$

Ex: Suppose that the equation $F(x,y)=0$ implicitly defines y as a function of x & defines x as a function of y . Show that $\frac{dy}{dx} \frac{dx}{dy} = 1$

$$\frac{dy}{dx} = - \frac{F_y}{F_x} \text{ \& } \frac{dx}{dy} = - \frac{F_y}{F_x} \Rightarrow \frac{dy}{dx} \frac{dx}{dy} = \left(- \frac{F_y}{F_x}\right) \left(- \frac{F_y}{F_x}\right) = 1$$

14.6) The Directional Derivative & the Gradient Vector

The gradient vector of the function $F(x, y)$ at (x_0, y_0) is defined by

$$\nabla F(x_0, y_0) = \langle F_x(x_0, y_0), F_y(x_0, y_0) \rangle$$

$$F(x, y, z) \Rightarrow \nabla F(x_0, y_0, z_0) = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0) \rangle$$

rate of change

The (Directional derivative) of the function $F(x, y)$ at the point (x_0, y_0) in the direction of the unit vector $\hat{v} = \langle a, b \rangle$ is

$$D_{\hat{v}} F(x_0, y_0) = \lim_{h \rightarrow 0} \frac{F(x_0 + ah, y_0 + bh) - F(x_0, y_0)}{h}$$

$$D_{\hat{v}} F(x_0, y_0) = \nabla F(x_0, y_0) \cdot \hat{v} \text{ (dot product)}$$

Ex: Find the directional derivative of the function F

$F(x, y) = x^2 y^3$ at the point $(-2, 3)$ in the direction $\vec{v} = 2\hat{i} - 5\hat{j}$

$$\nabla F = \langle F_x, F_y \rangle = \langle 2xy^3, 3x^2y^2 \rangle$$

$$\nabla F(-2, 3) = \langle -36, 108 \rangle$$

$$\text{unit vector} \Rightarrow \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\hat{i} - 5\hat{j}}{\sqrt{2^2 + 5^2}} = \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle$$

$$D_{\hat{v}} F(-2, 3) = \nabla F(-2, 3) \cdot \hat{v} = \langle -36, 108 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle =$$

$$\left\langle -36 \cdot \frac{2}{\sqrt{29}} + 108 \cdot \frac{-5}{\sqrt{29}} \right\rangle = \frac{-612}{\sqrt{29}}$$

Ex: Find the rate of change of the function $F(x,y) = \frac{x^2-y}{y^2}$ at the point $(2,1)$ in the direction indicated by the angle $\theta = \frac{\pi}{3}$

$$\nabla F = \langle F_x, F_y \rangle = \left\langle \frac{2x}{y^2}, \frac{y^2(-1) - (x^2-y)(2y)}{y^4} \right\rangle$$

$$\nabla F(2,1) = \langle 4, -7 \rangle$$

$$\vec{v} = \langle a, b \rangle : a = \cos \theta, b = \sin \theta$$

$$a = \cos \frac{\pi}{3} = \frac{1}{2}, b = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle, |\vec{v}| = 1$$

$$D_{\vec{v}} F(2,1) = \nabla F(2,1) \cdot \vec{v} = \langle 4, -7 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \frac{4 - 7\sqrt{3}}{2}$$

Ex: Find the rate of change of the function $F(x,y,z) = x^2 - 3yz^2$ at the point $P(2,-1,1)$ in the direction from P to the point $Q(3,1,\frac{1}{2})$

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, -3z^2, -9yz \rangle \Rightarrow \nabla F(2,-1,1) = \langle 4, -3, 9 \rangle$$

$$\vec{v} = \overrightarrow{PQ} = \langle Q - P \rangle = \langle 1, 2, -\frac{1}{2} \rangle$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 2, -\frac{1}{2} \rangle}{\sqrt{21}/2} = \left\langle \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \right\rangle$$

$$D_{\hat{v}} F(2,-1,1) = \nabla F(2,-1,1) \cdot \hat{v} = \langle 4, -3, 9 \rangle \cdot \left\langle \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \right\rangle$$

$$= \frac{8}{\sqrt{21}} - \frac{12}{\sqrt{21}} - \frac{9}{\sqrt{21}} = \frac{-13}{\sqrt{21}}$$

Ex: let $P(x,y) = \ln(x^2+2y) - 2\sqrt{x}$ Find

$$\lim_{h \rightarrow 0} \frac{P(4 - \frac{h}{3}, \frac{\sqrt{8}h}{3}) - P(4,0)}{h}$$

$$\langle -\frac{1}{3}, \frac{\sqrt{8}}{3} \rangle = 1 \Rightarrow \text{unit vector}$$

$$\lim_{h \rightarrow 0} \frac{P(4 - \frac{h}{3}, \frac{\sqrt{8}h}{3}) - P(4,0)}{h} = \nabla P(4,0) \cdot \langle -\frac{1}{3}, \frac{\sqrt{8}}{3} \rangle$$

$$\nabla P = \langle \frac{2x}{x^2+2y} + \frac{1}{\sqrt{x}}, \frac{2}{x^2+2y} \rangle \Rightarrow \nabla P(4,0) = \langle 0, \frac{1}{8} \rangle$$

$$\therefore \lim_{h \rightarrow 0} \frac{P(4 - \frac{h}{3}, \frac{\sqrt{8}h}{3}) - P(4,0)}{h} = \langle 0, \frac{1}{8} \rangle \cdot \langle -\frac{1}{3}, \frac{\sqrt{8}}{3} \rangle = \frac{\sqrt{8}}{24}$$

Ex: let $\hat{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$, $\hat{v} = \langle \frac{2\sqrt{2}}{3}, \frac{1}{3} \rangle$, $D_{\hat{u}}P(2,1) = 2$

and $D_{\hat{v}}P(2,1) = \frac{1}{3}$

(a) Find the gradient vector of P at the point $(2,1)$

let $\nabla P(2,1) = \langle a, b \rangle$

$$D_{\hat{u}}P(2,1) = 2 \Rightarrow \nabla P(2,1) \cdot \hat{u} = 2 \Rightarrow \langle a, b \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = 2$$

$$\frac{a-b}{\sqrt{2}} = 2 \Rightarrow a-b = 2\sqrt{2} \rightarrow (1)$$

$$D_{\hat{v}}P(2,1) = \frac{1}{3} \Rightarrow \nabla P(2,1) \cdot \hat{v} = \frac{1}{3} \Rightarrow \langle a, b \rangle \cdot \langle \frac{2\sqrt{2}}{3}, \frac{1}{3} \rangle = \frac{1}{3}$$

$$\frac{2\sqrt{2}a+b}{3} = \frac{1}{3} \Rightarrow 2\sqrt{2}a+b = 1 \rightarrow (2)$$

after solving $\Rightarrow \nabla P(2,1) = \langle a, b \rangle = \langle 1, 1-2\sqrt{2} \rangle$

(b) Find the directional derivative of P at the point $(2,1)$ in the direction of $\hat{i} - 2\hat{j}$

$$\nabla P(2,1) = \langle 1, 1-2\sqrt{2} \rangle$$

$$\vec{w} = \hat{i} - 2\hat{j} \Rightarrow |\vec{w}| = \sqrt{5} \Rightarrow \hat{w} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle \Rightarrow \text{unit vector}$$

$$D_{\hat{w}} P(2,1) = \nabla P(2,1) \cdot \hat{w} = \langle 1, 1-2\sqrt{2} \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle = \frac{4\sqrt{2}-1}{\sqrt{5}}$$

Since $\hat{i}, \hat{j}, \hat{k}$ are unit vectors then

$$D_{\hat{i}} P = P_x, \quad D_{\hat{j}} P = P_y, \quad D_{\hat{k}} P = P_z$$

Ex: use the Figure to estimate $D_{\vec{u}} P(2,2)$

$$D_{\vec{u}} P(2,2) = \nabla P(2,2) \cdot \vec{u} = |\nabla P| |\vec{u}| \cos \theta = |\nabla P(2,2)| \cos \theta = 3.7$$

unit vector = 1

$$\vec{u} = \langle \cos 150, \sin 150 \rangle = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

let $\hat{v} = \langle a, b \rangle$ be a unit vector then,

$$D_{\hat{v}} P(x_0, y_0) = \nabla P(x_0, y_0) \cdot \hat{v} = |\nabla P(x_0, y_0)| \cdot |\hat{v}| \cos \theta = |\nabla P(x_0, y_0)| \cos \theta$$

$$-1 \leq \cos \theta \leq 1 \Rightarrow -|\nabla P(x_0, y_0)| \leq D_{\hat{v}} P(x_0, y_0) \leq |\nabla P(x_0, y_0)|$$

$$-|\nabla P(x_0, y_0)| \leq D_{\hat{v}} P(x_0, y_0) \leq |\nabla P(x_0, y_0)|$$

The maximum value of the directional derivative $D_{\hat{v}} F(x_0, y_0)$ when $\hat{v}$ & ∇F are in the same direction.

$$D_{\hat{v}} F = |\nabla F|, \quad \hat{v} = \frac{\nabla F}{|\nabla F|}$$

The minimum value of the directional derivative $D_{\hat{v}} F(x_0, y_0)$ when $\hat{v}$ & $-\nabla F$ are in the same direction.

$$D_{\hat{v}} F = -|\nabla F|, \quad \hat{v} = -\frac{\nabla F}{|\nabla F|}$$

The maximum value of the directional derivative of F $\Leftrightarrow$ The maximum rate of change of F $\Leftrightarrow$ The Function F increases Fastest

The minimum value of the directional derivative of F $\Leftrightarrow$ The minimum rate of change of F $\Leftrightarrow$ The Function F decreases Fastest

Ex. Find the max. dire. der of the function $F(x, y) = 2y^2 \sqrt{x}$ at the point $(9, -3)$ and Find the direction

$$\nabla F = \left\langle \frac{y^2}{\sqrt{x}}, 4y\sqrt{x} \right\rangle \Rightarrow \nabla F(9, -3) = \langle 3, -36 \rangle$$

$$\text{The max. dir. der} = |\nabla F(9, -3)| = \sqrt{3^2 + 36^2} = \sqrt{1305}$$

$$\text{The direction } \nabla F(9, -3) = \langle 3, -36 \rangle = \langle 1, -12 \rangle$$

Find the direction in which the function $F(x, y, z) = xe^{x-yz}$ decreases fastest at the point $(2, 1, 2)$

$$\nabla F = \langle xe^{x-yz} + e^{x-yz}, -xz e^{x-yz}, -xy e^{x-yz} \rangle$$

$$\nabla F(2, 1, 2) = \langle 3, -4, -2 \rangle \Rightarrow -\nabla F(2, 1, 2) = \langle -3, 4, 2 \rangle$$

Ex: Find the unit vector $\hat{u}$ if $\nabla F(1, 2) = \langle 3, -4 \rangle$ & $D_{\hat{u}} F(1, 2) = 5$

$$|\nabla F(1, 2)| = \sqrt{9+16} = 5 \Rightarrow D_{\hat{u}} F(1, 2) = |\nabla F(1, 2)| = 5$$

$D_{\hat{u}} F(1, 2)$ has its max value $\Rightarrow \hat{u}$ & ∇F are in the same direction

$$\hat{u} = \frac{\nabla F(1, 2)}{|\nabla F|} = \frac{\langle 3, -4 \rangle}{5} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

(إذا كانت -5 موجه $-\nabla F$)

إذا كان الرقم خارج القوس $\leftarrow$ لا يجب unit vector

Let $S: F(x, y, z) = k$ be a surface, $P(x_0, y_0, z_0)$ be a point on S and let $\nabla F(x_0, y_0, z_0) = \langle a, b, c \rangle$ Then

① The equation of the tangent plane to the surface S at P is
 $ax + by + cz = ax_0 + by_0 + cz_0$

② The parametric eq of the normal line to the surface S at P are
 $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$

Ex. Find the eq of the tangent plane & the normal line at the point $(-2, 1, 3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$

$$\text{The eq} \Rightarrow \frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3 = 0$$

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3$$

$$\nabla F = \left\langle \frac{x}{2}, 2y, \frac{2z}{9} \right\rangle \Rightarrow \nabla F(-2, 1, 3) = \left\langle -1, 2, \frac{2}{3} \right\rangle$$

$\Rightarrow$ The eq of the tangent plane at $(-2, 1, 3)$ is

$$-x + 2y + \frac{2}{3}z = 2 + 2 + 2 = 6$$

$$-3x + 6y + 2z = 18$$

$\Rightarrow$ The eq of the normal line at $(-2, 1, 3)$ is

$$(x = -2 - t, y = 1 + 2t, z = 3 + \frac{2}{3}t)$$

$$x = -2 - 3t, y = 1 + 6t, z = 3 + 2t \quad \leftarrow \left\langle -1, 2, \frac{2}{3} \right\rangle$$

Ex. Find the eq of the tangent plane & the normal line at the point $(-2, 1, 3)$ to the surface $z = x^2 + y^2$

$$\text{The eq} \Rightarrow x^2 + y^2 - z = 0$$

$$F(x, y, z) = x^2 + y^2 - z$$

$$\nabla F = \langle 2x, 2y, -1 \rangle \Rightarrow \nabla F(-2, 1, 3) = \langle -4, 2, -1 \rangle$$

$\Rightarrow$ The eq of the tangent plane at $(-2, 1, 3)$

$$-4x + 2y - z = 7$$

$\Rightarrow$ The eq of the normal line at $(-2, 1, 3)$

$$x = -2 - 4t, y = 1 + 2t, z = 3 - t$$

Ex: At what point the surface $y = x^2 + z^2$ is tangent to the plane parallel to the plane $x + 2y + 3z = 1$

$$x^2 + z^2 - y = 0 \Rightarrow F(x, y, z) = x^2 + z^2 - y$$

$$\nabla F = \langle 2x, -1, 2z \rangle$$

$$\text{plane: } x + 2y + 3z = 1 \Rightarrow \hat{v} = \langle 1, 2, 3 \rangle$$

$$\langle 2x, -1, 2z \rangle \parallel \langle 1, 2, 3 \rangle$$

$$\frac{2x}{1} = \frac{-1}{2} = \frac{2z}{3} \xrightarrow{\text{surface is a paraboloid}} x = -\frac{1}{4}, z = -\frac{3}{4}, y = \frac{10}{16}$$

Ex: At what point the surface $x^2 - y^2 + z^2 - 2x = 1$ has a normal line parallel to the line $x = 4t, y = 1 - 2t, z = 2t$

$$x^2 - y^2 + z^2 - 2x = 1 \Rightarrow x^2 - y^2 + z^2 - 2x - 1 = 0$$

$$F(x, y, z) = x^2 - y^2 + z^2 - 2x - 1 \Rightarrow \nabla F = \langle 2x - 2, -2y, 2z \rangle$$

$$\text{normal line} \parallel \text{line } x = 4t / y = 1 - 2t / z = 2t$$

$$\langle 2x - 2, -2y, 2z \rangle \parallel \langle 4, -2, 2 \rangle$$

$$\frac{2x - 2}{4} = \frac{-2y}{-2} = \frac{2z}{2} \Rightarrow x = 0, y = -\frac{1}{2}, z = \frac{1}{2}$$

Ex: At what point does the normal line through the point $(1, 1, 2)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 21$ intersect the sphere $x^2 + y^2 + z^2 = 4$

$$4x^2 + y^2 + 4z^2 = 12 \Rightarrow 4x^2 + y^2 + 4z^2 - 12 = 0$$

$$F(x, y, z) = 4x^2 + y^2 + 4z^2 - 12 \Rightarrow \nabla F = \langle 8x, 2y, 8z \rangle$$

$$\nabla F(1, 1, 2) = \langle 8, 2, 16 \rangle \div 2$$

$$\langle 4, 1, 8 \rangle \parallel \text{normal line}$$

$\Rightarrow$ The eqs of the normal line are

$$x = 1 + 4t \quad / \quad y = 1 + t \quad / \quad z = 2 + 8t$$

The normal line intersects the sphere $x^2 + y^2 + z^2 = 4$

substitute $(x = 1 + 4t, y = 1 + t, z = 2 + 8t)$ in the eq $x^2 + y^2 + z^2 = 4$

$$(1 + 4t)^2 + (1 + t)^2 + (2 + 8t)^2 = 4$$

$$1 + 8t + 16t^2 + 1 + 2t + t^2 + 4 + 32t + 64t^2 = 4$$

$$t = 0 \text{ or } t = -\frac{42}{81}$$

$$\text{when } t = 0 \Rightarrow x = 1, y = 1, z = 2$$

$$\text{when } t = -\frac{42}{81} \Rightarrow x = -\frac{87}{81}, y = \frac{39}{81}, z = -\frac{174}{81}$$

The points are $(1, 1, 2)$ & $(-\frac{87}{81}, \frac{39}{81}, -\frac{174}{81})$

Ex: Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$ intersects the paraboloid a second time

$$z = x^2 + y^2 \Rightarrow x^2 + y^2 - z = 0 \Rightarrow F(x, y, z) = x^2 + y^2 - z$$

$$\nabla F = \langle 2x, 2y, -1 \rangle \Rightarrow \nabla F(1, 1, 2) = \langle 2, 2, -1 \rangle$$

$\Rightarrow$ The eq of the normal line are $x = 1 + 2t$ / $y = 1 + 2t$ / $z = 2 - t$

The normal line intersects the paraboloid $z = x^2 + y^2$
substitute $(x = 1 + 2t, y = 1 + 2t, z = 2 - t)$ in the eq $z = x^2 + y^2$

$$2 - t = (1 + 2t)^2 + (1 + 2t)^2 \Rightarrow 2 - t = 2(1 + 4t + 4t^2)$$

$$8t^2 + 9t = 0 \Rightarrow t(8t + 9) = 0 \Rightarrow t = 0 \text{ or } t = -\frac{9}{8}$$

$$\text{when } t = 0 \Rightarrow x = 1, y = 1, z = 2$$

$$\text{when } t = -\frac{9}{8} \Rightarrow x = -\frac{10}{8}, y = -\frac{10}{8}, z = \frac{25}{8}$$

The point are $(1, 1, 2)$ & $(-\frac{10}{8}, -\frac{10}{8}, \frac{25}{8})$

14.7) Maximum & Minimum Values

Def 1: A function $F(x,y)$ is said to have,

- 1) a local maximum value at a point $(a,b) \in \text{Dom } F$ if $F(a,b) \geq F(x,y)$ for all $(x,y) \in D$
- 2) a local minimum value at a point $(a,b) \in \text{Dom } F$ if $F(a,b) \leq F(x,y)$ for all $(x,y) \in D$
- 3) an absolute maximum value at a point $(a,b) \in \text{Dom } F$ if $F(a,b) \geq F(x,y)$ for all $(x,y) \in D$
- 4) an absolute minimum value at a point $(a,b) \in \text{Dom } F$ if $F(a,b) \leq F(x,y)$ for all $(x,y) \in D$
- 5) a local extremum at a point (a,b) if F has a local maximum or minimum value at (a,b)
- 6) an absolute extremum at a point (a,b) if F has an absolute maximum & minimum value at (a,b)

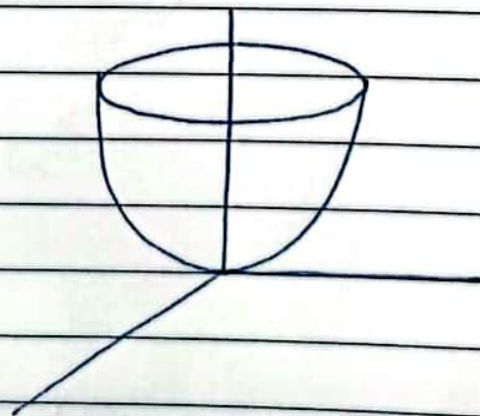
Ex: Find the absolute & local extrema of the function

$$F(x,y) = 2x^2 + y^2$$

$z = 2x^2 + y^2$, paraboloid

$\Rightarrow F$ has local minimum value at $(0,0)$
the local minimum value $F(0,0) = 0$

$\Rightarrow F$ has absolute minimum value at $(0,0)$
the absolute minimum value $F(0,0) = 0$



~~#~~ Part.1 = Local Maximum & Minimum Values

Def. A function $F(x,y)$ is said to have critical point at $(a,b) \in \text{Dom } F$ if

① $F_x(a,b) = 0$ & $F_y(a,b) = 0$

or

② $F_x(a,b)$ does not exist, $F_y(a,b)$ does not exist

Ex: Find the value of a & b that makes the function F has a critical point at $(1, -1)$ where $F(x,y) = x^2y + 3axy^2 - bxy$

$$F_x = 2xy + 3ay^2 - by \quad \& \quad F_y = x^2 + 6axy - bx$$

$$(1, -1) \text{ is a critical point} : F_x(1, -1) = 0, F_y(1, -1) = 0$$

$$F_x(1, -1) = 0 \Rightarrow -2 + 3a + b = 0 \Rightarrow 3a + b = 2 \rightarrow \textcircled{1}$$

$$F_y(1, -1) = 0 \Rightarrow 1 - 6a - b = 0 \Rightarrow 6a + b = 1 \rightarrow \textcircled{2}$$

$$\therefore a = -\frac{1}{3}, b = 1$$

$$\# D(a,b) = F_{xx}(a,b) F_{yy}(a,b) - (F_{xy}(a,b))^2$$

① IF $D(a,b) > 0$ & $F_{xx}(a,b) > 0 \rightarrow$ local minimum

② IF $D(a,b) > 0$ & $F_{xx}(a,b) < 0 \rightarrow$ local maximum

③ IF $D(a,b) < 0 \rightarrow$ neither a local maximum nor minimum

In this case we say that F has a saddle point at (a,b)

Ex: Find & classify the critical points of the function f as local maximum, local minimum or saddle point where $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$. Moreover Find the local maximum & minimum values of f .

$$P_x = 6x^2 + 6y^2 - 150 \quad \& \quad P_y = 12xy - 9y^2$$

$$0 = 6x^2 + 6y^2 - 150$$

$$0 = 12xy - 9y^2$$

$$6x^2 + 6y^2 = 150$$

$$4xy - 3y^2 = 0 \rightarrow (2)$$

$$x^2 + y^2 = 25 \rightarrow (1)$$

$$y(4x - 3y) = 0$$

$$y = 0 \text{ or } y = \frac{4}{3}x$$

$$(1) \text{ if } y = 0 \Rightarrow x = \pm 5$$

$$(2) \text{ if } y = \frac{4}{3}x \Rightarrow x = \pm 3 \Rightarrow x = 3 \rightarrow y = 4 / x = -3 \rightarrow y = -4$$

the critical points $(5,0)$ $(-5,0)$ $(3,4)$ $(-3,-4)$

$$P_{xx} = 12x \quad \& \quad P_{yy} = 12x - 18y \quad \& \quad P_{xy} = 12y$$

$$D = P_{xx} P_{yy} - (P_{xy})^2 = 12x(12x - 18y) - (12y)^2$$

$$\text{At } (-5,0) \Rightarrow D(-5,0) = 12(-5)(12(-5)) \geq 0 \quad P_{xx}(-5,0) = 12(-5) < 0$$

local maximum at the point $(-5,0)$

$$f(-5,0) = 500$$

$$\text{At } (5,0) \Rightarrow D(5,0) = 12(5) \cdot 12(5) > 0 \quad \& \quad P_{xx}(+5,0) = 12(5) > 0$$

local minimum at the point $(5,0)$ $f(5,0) = -500$

$$\text{At } (-3,-4) \Rightarrow D(-3,-4) = 12(-3)(12(-3) - 18(-4) - (12(-3))^2) < 0$$

saddle point at $(-3,-4)$

$$\text{At } (3,4) \Rightarrow D(3,4) = 12(3)(12(3) - 18(4) - (12(4))^2) < 0$$

saddle point

Ex: Find the local maximum & local minimum value of the function $F(x,y) = x^4 + y^4 - 4xy + 1$

$$F_x = 4x^3 - 4y \quad \& \quad F_y = 4y^3 - 4x$$

$$0 = 4x^3 - 4y \quad \quad 0 = 4y^3 - 4x$$

$$y = x^3 \quad \quad x = y^3$$

$$\therefore x = x^9 \rightarrow x = 0, -1, 1$$

$$\textcircled{1} \text{ IF } x=0 \rightarrow y=0$$

$$\textcircled{2} \text{ IF } x=-1 \rightarrow y=-1$$

$$\textcircled{3} \text{ IF } x=1 \rightarrow y=1$$

The critical point $(0,0), (-1,-1), (1,1)$

$$F_{xx} = 12x^2 \quad \& \quad F_{yy} = 12y^2 \quad \& \quad F_{xy} = -4$$

$$D = F_{xx}F_{yy} - (F_{xy})^2 \Rightarrow D = 12x^2(12y^2) - (-4)^2$$

At $(0,0)$: $D(0,0) = -16 < 0 \rightarrow$ saddle point at $(0,0)$

At $(-1,-1)$: $D(-1,-1) = 12(12) - 16 > 0$ & $F_{xx}(-1,-1) = 12 > 0$

local minimum at $(-1,-1)$ $F(-1,-1) = -1$

At $(1,1)$: $D(1,1) = 12(12) - 16 > 0$ & $F_{xx}(1,1) = 12 > 0$

local minimum at $(1,1)$ $F(1,1) = -1$

Part 2: Absolute Max & Mini Values

Let $f(x,y)$ be with $D \subset \mathbb{R}^2$ & has only one critical point at (a,b)

- ① If $f(a,b)$ is a local maximum value of the function f then $f(a,b)$ is an absolute maximum value of f
- ② If $f(a,b)$ is a local minimum value of the function f then $f(a,b)$ is an absolute minimum value of f

Ex. Find the absolute max & min value of the function
 $f(x,y) = x^2 + y^2 - 2x - 6y + 12$

Ex. Find the shortest distance from the point $(1,0,-2)$
to the plane $x + 2y + z = 4$

(Part 3) Absolute Max & Min values of Function over closed bounded sets.

Def: If $F(x, y)$ is cont on a closed, bounded set D in $\mathbb{R}^2$ then it attains an absolute maximum value $F(x_1, y_1)$ & an absolute minimum value $F(x_2, y_2)$ at some points (x_1, y_1) & (x_2, y_2) ~~W.D~~

To Find the absolute max & min values of a continuous function $F(x, y)$ on a closed, bounded set D .

- ① Find the value of the critical points of $F(x, y)$ in D
- ② Find the extreme values of $F(x, y)$ on the boundary of D
- ③ The largest of the values from step 1 & 2 is the absolute max and the smallest is the absolute minimum

15.1) Double Integrals over Rectangles

15.2) Iterated Integrals

The Rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ can be expressed as $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d]$

$\iint_R f(x, y) dA$ can be expressed as an iterated integral as $\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$

If $f(x, y) \geq g(x, y)$ then $\iint_R f dA \geq \iint_R g dA$

If the variables x & y in $f(x, y)$ are separated that is $f(x, y) = g(x)h(y)$ then

$$\iint_R f dA = \int_a^b \int_c^d g(x)h(y) dy dx = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right)$$

Ex: Evaluate the double integral $\iint_R (x - 3y^2) dA$ where $R = \{(x, y) : 0 \leq x \leq 2, -1 \leq y \leq 3\}$

$$\begin{aligned} \iint_R x - 3y^2 dA &= \int_{-1}^3 \int_0^2 (x - 3y^2) dx dy = \int_{-1}^3 \left(\frac{x^2}{2} - 3y^2 x \right) \bigg|_0^2 dy \\ &= \int_{-1}^3 (2 - 6y^2) dy = 2y - 2y^3 \bigg|_{-1}^3 = (6 - 54) - (-2 + 2) = -48 \end{aligned}$$

Ex: Compute the following double integral $\iint_R x \sin(xy) dA$ where $R = [1, 2] \times [0, \frac{\pi}{2}]$

$$\begin{aligned} \iint_R x \sin xy dA &= \int_1^2 \int_0^{\frac{\pi}{2}} x \sin xy dy dx = \int_1^2 x \cdot \left(\frac{-\cos xy}{x} \right) \bigg|_0^{\frac{\pi}{2}} dx \\ &= \int_1^2 (\cos \frac{\pi}{2} x - 1) dx = \left(\frac{\sin \frac{\pi}{2} x}{\frac{\pi}{2}} - x \right) \bigg|_1^2 = 1 + \frac{2}{\pi} \end{aligned}$$

Recall that $y = \sqrt{r^2 - (x-a)^2}$ is the eq of semicircle.

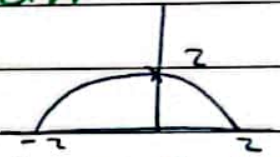
$$\Rightarrow \int_{a-r}^{a+r} \sqrt{r^2 - (x-a)^2} dx = \frac{1}{2} \pi r^2$$

$$\Rightarrow \int_a^{a+r} \sqrt{r^2 - (x-a)^2} dx = \frac{1}{4} \pi r^2$$

$$\Rightarrow \int_{a-r}^a \sqrt{r^2 - (x-a)^2} dx = \frac{1}{4} \pi r^2$$

Ex: Evaluate the double integral $\iint_R 4 - x^2 dA$
where $R = [-2, 2] \times [0, 3]$

$$\int_{-2}^2 \int_0^3 \sqrt{4 - x^2} dy dx = 3 \int_{-2}^2 \sqrt{4 - x^2} dx = 3 \times \frac{1}{2} \pi \times 4 = 6\pi$$



Ex: Evaluate the iterated integral $\int_0^\pi \int_0^{\frac{\pi}{2}} \sin \frac{x}{3} \cos 2y dy dx$

$$\left(\int_0^\pi \sin \frac{x}{3} dx \right) \left(\int_0^{\frac{\pi}{2}} \cos 2y dy \right) = -3 \cos \frac{x}{3} \Big|_0^\pi \times \sin 2y \Big|_0^{\frac{\pi}{2}} = \frac{3}{8}$$

Ex: Evaluate the iterated integral $\int_0^2 \int_0^1 f(x, y) dy dx$

where $f(x, y) = \begin{cases} 2y, & x \geq e^y \\ 4x, & x < e^y \end{cases}$

$$\begin{aligned} \int_0^2 \int_0^1 f(x, y) dy dx &= \int_0^1 \int_0^2 f(x, y) dx dy = \int_0^1 \left(\int_0^{e^y} f dx + \int_{e^y}^2 f dx \right) dy \\ &= \int_0^1 \left(\int_0^{e^y} 4x dx + \int_{e^y}^2 2y dx \right) dy = \int_0^1 \left(2x^2 \Big|_0^{e^y} + (2 - e^y) 2y \right) dy \\ &= \int_0^1 (2e^{2y} + 4y - 2ye^y) dy = e^{2y} + 2y^2 - (2ye^y - 2e^y) \Big|_0^1 \\ &= e + 1 \end{aligned}$$

15.3) Double Integrals over General Regions

Ex: Sketch the region & change the order of integration

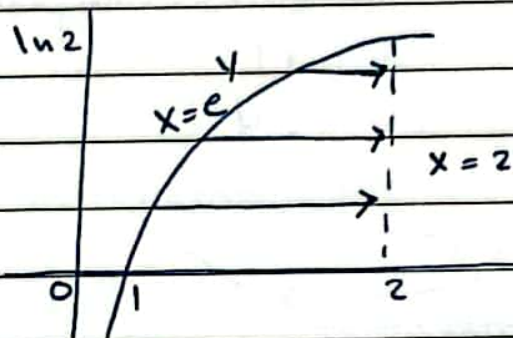
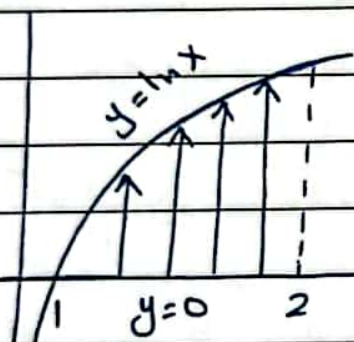
1) $\int_1^2 \int_0^{\ln x} f(x,y) dy dx \Rightarrow$ إذا كانت حدود التكامل فيها ليست نقطة وخط
 مع فغير لا تقبل وخط

① $Ad = dy dx \Rightarrow$ Type ①

② $Ad = dx dy \Rightarrow$ Type ②

$y=0 \rightarrow y=\ln x$
 $x=1 \rightarrow x=2$

$x=e^y \rightarrow x=2$
 $y=0 \rightarrow y=\ln 2$



$$\int_1^2 \int_0^{\ln x} f(x,y) dy dx = \int_0^{\ln 2} \int_{e^y}^2 f(x,y) dx dy$$

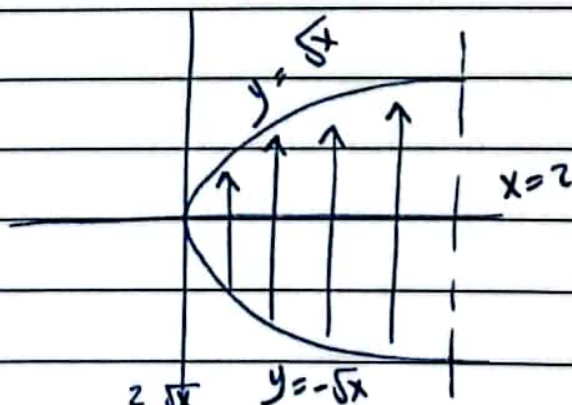
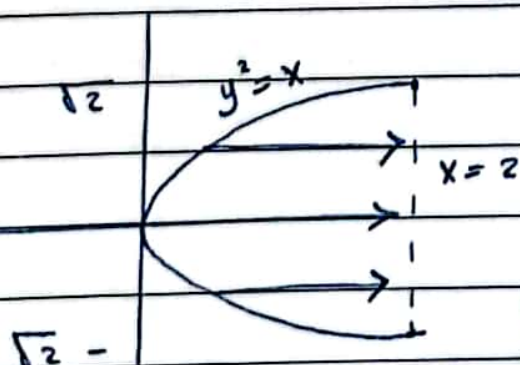
2) $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^2 f(x,y) dx dy$

① $Ad = dx dy \Rightarrow$ Type ②

② $Ad = dy dx \Rightarrow$ Type ①

$x=y^2 \rightarrow x=2$
 $y=-\sqrt{2} \rightarrow y=\sqrt{2}$

$y=-\sqrt{x} \rightarrow y=\sqrt{x}$
 $x=0 \rightarrow x=2$



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^2 f(x,y) dx dy = \int_0^2 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y) dy dx$$

Ex: change the order of integration & sketch it

$$\int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} f(x,y) dx dy$$

$$dA = dx dy \rightarrow \text{Type (2)}$$

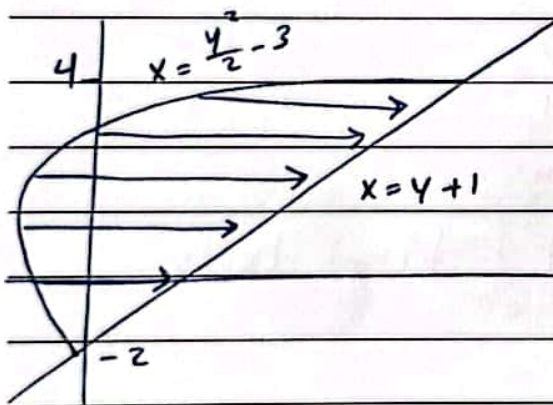
$$x = \frac{y^2}{2} - 3 \rightarrow x = y + 1$$

$$y = -2 \rightarrow y = 4$$

To find the intersection points

$$\frac{y^2}{2} - 3 = y + 1$$

$$y = -2 \text{ or } y = 4$$



$$dA = dy dx \rightarrow \text{Type (1)}$$

$$y = \pm \sqrt{2x+6}$$

$$y = -\sqrt{2x+6} \rightarrow y = \sqrt{2x+6}$$

$$y = -2 \rightarrow x = -1$$

$$y = 4 \rightarrow x = 5$$

we have 2 region

Region 1

$$y = -\sqrt{2x+6} \rightarrow y = \sqrt{2x+6}$$

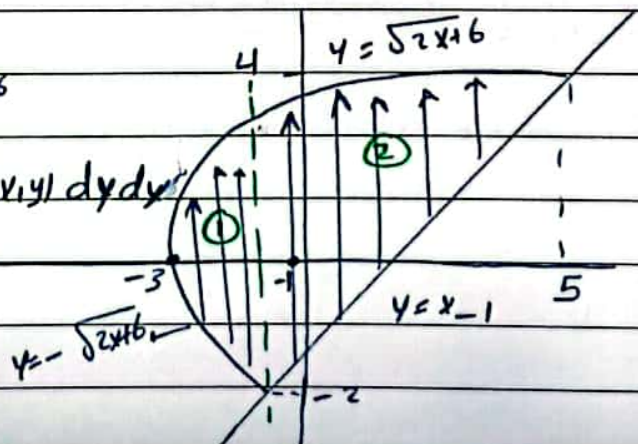
$$\text{vertical lines } x = -3 \rightarrow x = -1$$

Region 2

$$y = x - 1 \rightarrow y = \sqrt{2x+6}$$

$$x = -1 \rightarrow x = 5$$

$$\int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} f(x,y) dx dy = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y) dy dx + \int_{-1}^5 \int_{x-1}^{\sqrt{2x+6}} f(x,y) dy dx$$



Ex. Evaluate $\iint_D xy \, dA$, where D is the shaded region in the Figure

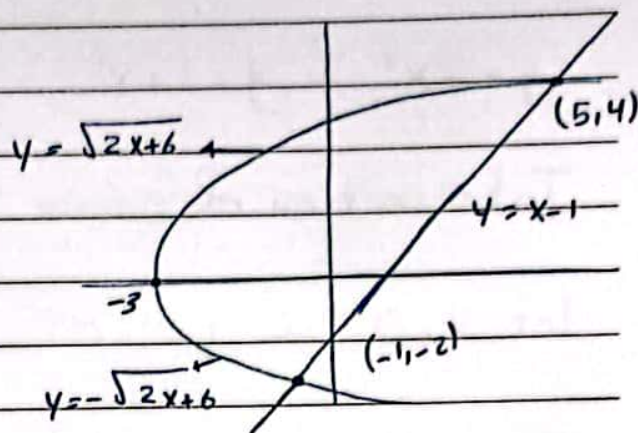
Type 2 is better than type 1

$$x = \frac{y^2}{2} - 3 \text{ \& } x = y + 1$$

Take $dA = dx dy$

$$\iint_D xy \, dA = \int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} xy \, dx \, dy$$

$$= 36$$



Ex. Evaluate $\iint_D xy \, dA$ where D is the region bounded by
 $y = x - 1$ ① & $y^2 = 2x + 6$ ②

$$y = x - 1 \Rightarrow x = y + 1$$

$$y^2 = 2x + 6 \Rightarrow x = \frac{y^2 - 6}{2}$$

$$(dA = dx dy)$$

1 kind of dA

عدد الخيارات 2 $dy dx$

عدد الخيارات 3 $dx dy$ x موضع القاسم

$$\text{Intersection of curves: } y + 1 = \frac{y^2}{2} - 3 \Rightarrow y = -2 \text{ or } y = 4$$

$$-2 \leq y \leq 4$$

$$\text{let } y = 0 \Rightarrow 1) 0 = x - 1$$

$$x = 1$$

$$2) 0 = 2x + 6$$

$$x = -3$$

3 الخيارات أكبر الخيارات والأعلى

نأخذ نقطة اختبار من

الفترة $[-2, 4]$

ونعوضها بالمختصين والذي

أكون مقبولة أصغر يكون

الحد الأدنى والذي مقبولة

أكبر يكون الحد الأعلى

$$\iint_D xy \, dA = \int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} xy \, dx \, dy = \int_{-2}^4 \frac{x^2}{2} y \Big|_{\frac{y^2}{2}-3}^{y+1} dy$$

$$= 36$$

Ex. Compute $\iint_D (x+2y) dA$ where D is the region enclosed by $y=2x^2$ & $y=1+x^2$

$$y=2x^2 \text{ \& } y=1+x^2 \Rightarrow \text{Type I } dydx$$

Intersection of curves: $2x^2 = 1+x^2 \Rightarrow x^2 = 1 \Rightarrow -1 \leq x \leq 1$

$$\text{let } x=0 \Rightarrow 1) y=0$$

$$2) y=1$$

$$\iint_D (x+2y) dA = \int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y) dy dx = \frac{32}{15}$$

Ex. Compute the following iterated integral

$$1) \int_0^1 \int_{2y}^2 e^{x^2} dx dy$$

Type 2

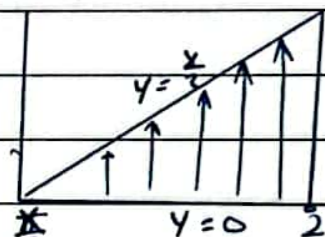
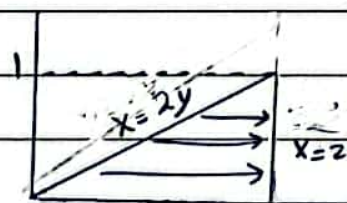
$$x=2y \rightarrow x=2$$

$$y=0 \rightarrow y=1$$

Type 1

$$y=0 \rightarrow y=\frac{x}{2}$$

$$x=0 \rightarrow x=2$$



$$\Rightarrow \int_0^1 \int_{2y}^2 e^{x^2} dx dy = \int_0^2 \int_0^{\frac{x}{2}} e^{x^2} dy dx = \frac{e^4 - 1}{4}$$

$$2) \int_0^{\frac{1}{2}} \int_{2x}^1 \sin(y^2) dy dx$$

Type 1

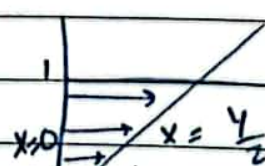
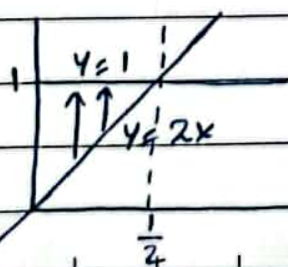
$$y=2x \rightarrow y=1$$

$$x=0 \rightarrow x=\frac{1}{2}$$

Type 2

$$x=0 \rightarrow x=\frac{y}{2}$$

$$y=0 \rightarrow y=1$$



$$\int_0^{\frac{1}{2}} \int_{2x}^1 \sin(y^2) dx dy =$$

$$\frac{1 - \cos 1}{4}$$

#let D be a region in the xy -plane. Then the area of $D = \iint_D 1 \, dA$

Ex: Find the area of the shaded region in the figure

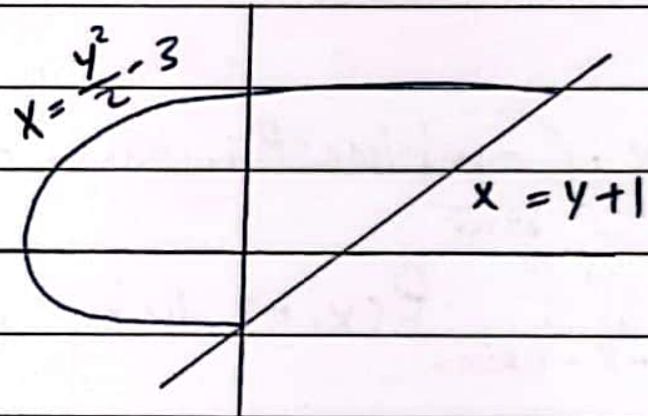
Type 2 $dx dy$

First we find the point of intersection

$$\frac{y^2}{2} - 3 = y + 1 \Rightarrow y^2 - 2y - 8 = 0$$

$$(y-4)(y+2) = 0$$

$$y = -2 \text{ or } y = 4$$



$$\text{Area} = \iint_D 1 \, dA = \int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} 1 \, dx dy = \int_{-2}^4 (y+1 - \frac{y^2}{2} + 3) dy =$$

Ex: Find the area of the region bounded by the curves $y = e^{2x}$, $y = 2$, $x = 4$

$$y = e^{2x}, y = 2$$

$$x = 4 \rightarrow x = ?? \text{ From intersection } e^{2x} = 2 \Rightarrow x = \frac{\ln 2}{2}$$

$$x = \frac{\ln 2}{2} \rightarrow x = 4 \rightarrow \text{from } \frac{\ln 2}{2} \text{ to } 4 \rightarrow e^{2x} \text{ dx}$$

$$\text{Area} = \int_{\frac{\ln 2}{2}}^4 \int_2^{e^{2x}} 1 \, dy dx = \int_{\frac{\ln 2}{2}}^4 (e^{2x} - 2) dx =$$

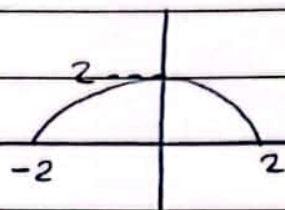
Ex: Find the Following

$$1) \int_{-2}^2 \int_0^{\sqrt{4-x^2}} -3 dy dx = -3 \int_{-2}^2 \int_0^{\sqrt{4-x^2}} 1 dy dx = \text{Area}$$

$$y=0 \rightarrow y=\sqrt{4-x^2}$$

$$x=-2 \rightarrow x=2$$

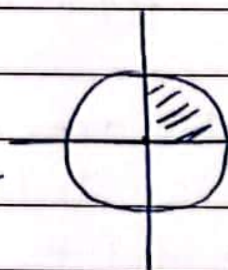
$$\int_{-2}^2 \int_0^{\sqrt{4-x^2}} -3 dy dx = -3 \int_{-2}^2 \int_0^{\sqrt{4-x^2}} 1 dy dx = -3 \text{Area} = -6\pi$$



$$2) \iint_D 2dA \text{ where } D = \{(x,y) : x^2 + y^2 \leq 9, \underline{x \geq 0}, \underline{y \geq 0}\}$$

$$\iint_D 2dA = 2 \iint_D 1 dA = 2 \text{Area}$$

$$= 2 \cdot \frac{1}{4} \pi \cdot 4 = 2\pi$$



Ex: Combine the sum of the two double integral as a single

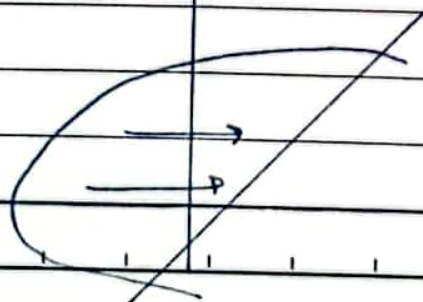
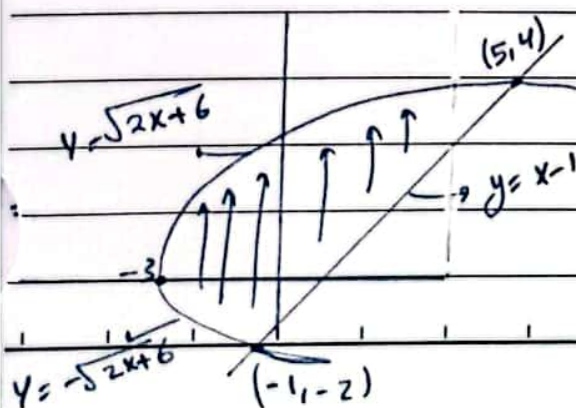
$$\int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y) dy dx + \int_{-1}^5 \int_{x-1}^{\sqrt{2x+6}} f(x,y) dy dx$$

$$\int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y) dy dx = y = -\sqrt{2x+6} \rightarrow y = \sqrt{2x+6}$$

$$x = -3 \rightarrow x = -1$$

$$\int_{-1}^5 \int_{x-1}^{\sqrt{2x+6}} f(x,y) dy dx = y = x-1 \rightarrow y = \sqrt{2x+6}$$

$$x = -1 \rightarrow x = 5$$



$$\int_{-2}^4 \int_{\frac{y^2}{2}-3}^{y+1} f(x,y) dx dy$$

Five Apple

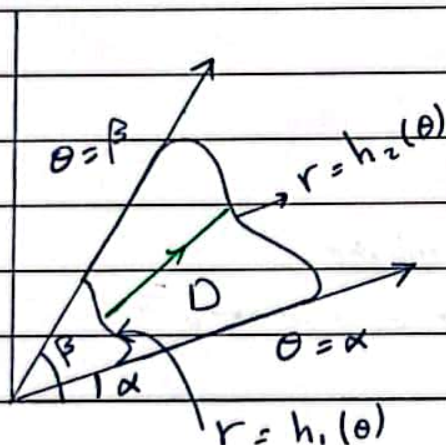
let D be a closed bounded region in the xy -plane such that $m \leq f(x,y) \leq M$ for all (x,y) in D . Then

$$m(\text{Area}) \leq \iint_D f dA \leq M(\text{Area})$$

Ex: Estimate the value of $\iint_D e^{\sin x \cos y} dA$ where D is the region enclosed by the disk $x^2 + y^2 \leq 4$ → circle

$$\begin{aligned} -1 \leq \sin x \cos y \leq 1 &\Rightarrow e^{-1} \leq e^{\sin x \cos y} \leq e \\ \text{Area of } D = 4\pi &\Rightarrow 4\pi e^{-1} \leq \iint_D e^{\sin x \cos y} dA \leq 4\pi e \end{aligned}$$

15.4) Double Integral in Polar Coordinates



$$\text{let } D = \{(r, \theta) : h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta\}$$

$$\text{where } 0 < \beta - \alpha \leq 2\pi$$

Then

$$\iint_D f(x,y) dA =$$

$$\int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

تذكر

معرفة r نقطة معينة

معرفة البداية ونهاية

خارج الدائرة أو داخلها

إذا دخلت المنطقة يكون البداية

زاوية الانحناء في محور السينات الموجب

منحنية الانحناء

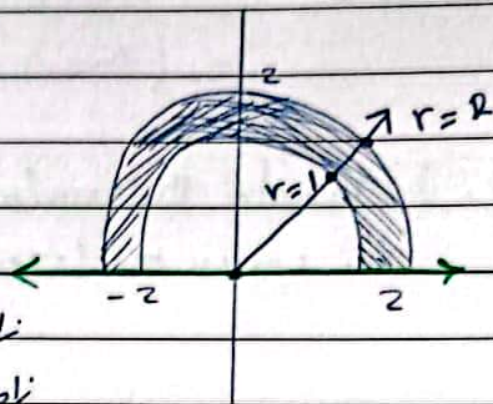
Ex: Evaluate $\iint_D (3x + 4y^2) dA$ where D is the region in the (upper half-plane) bounded by $x^2 + y^2 = 1$ & $x^2 + y^2 = 4$

$$x^2 + y^2 = 1 \Rightarrow r = 1$$

$$x^2 + y^2 = 4 \Rightarrow r = 2$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$



من 0 إلى 2π
من 0 إلى π

$$\begin{aligned} \iint_D (3x + 4y^2) dA &= \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta = \\ &= \int_0^\pi \int_1^2 (3r^2 \cos \theta + 4r^3 \sin^2 \theta) dr d\theta = \int_0^\pi \left[r^3 \cos \theta + r^4 \sin^2 \theta \right]_1^2 d\theta \\ &= \int_0^\pi 7 \cos \theta + 15 \sin^2 \theta d\theta = 7 \sin \theta + 15 \left(\frac{\theta}{2} - \frac{1}{2} \sin \theta \right) \Big|_0^\pi = \boxed{15\pi} \end{aligned}$$

Ex: Evaluate $\iint_D \arctan \frac{y}{x} dA$ where

$$D = \{(x, y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq \sqrt{3}x\}$$

$$x^2 + y^2 = 1 \rightarrow r = 1$$

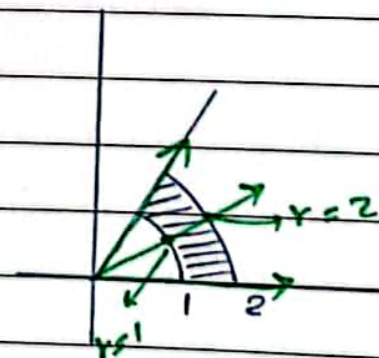
$$x^2 + y^2 = 4 \rightarrow r = 2$$

$$\tan^{-1} \frac{y}{x} = \theta$$

$$y = 0 \rightarrow \frac{y}{x} = 0$$

$$\theta = 0 \rightarrow \tan^{-1} \sqrt{3} = \frac{\pi}{3}$$

$$\theta = 0 \rightarrow \theta = \frac{\pi}{3}$$

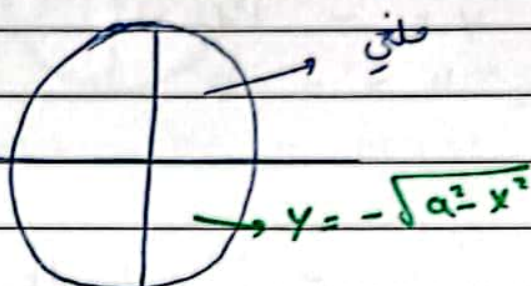
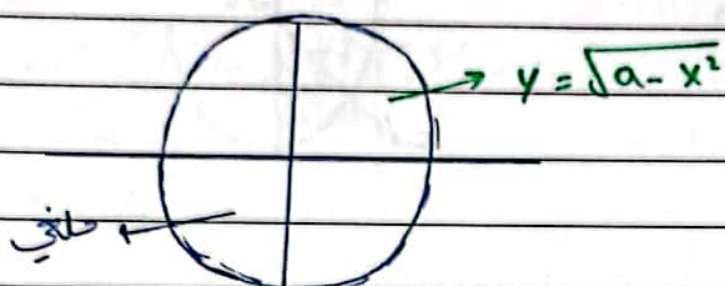


$$\begin{aligned} \iint_D \arctan \frac{y}{x} dA &= \iint_D \tan^{-1} \frac{y}{x} dA = \int_0^{\frac{\pi}{3}} \int_1^2 \theta r dr d\theta \\ &= \left(\int_0^{\frac{\pi}{3}} \theta d\theta \right) \left(\int_1^2 r dr \right) = \frac{3\pi^2}{36} \end{aligned}$$

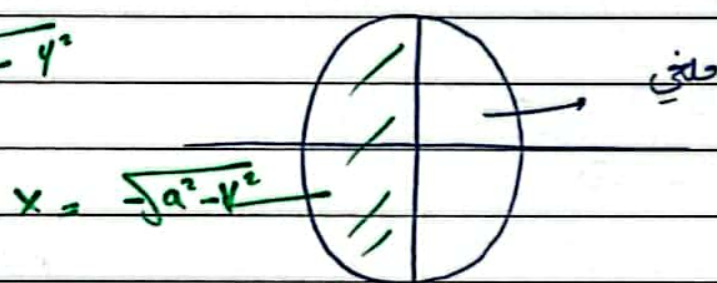
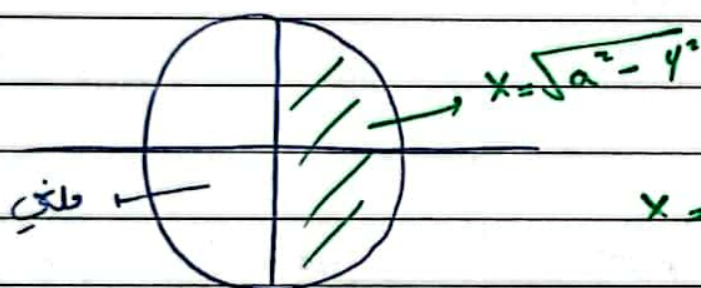
let $a > 0$

① $x^2 + y^2 = a^2 \Rightarrow r = a$

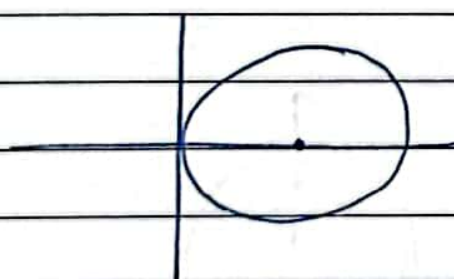
$x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2} \text{ or } y = -\sqrt{a^2 - x^2}$



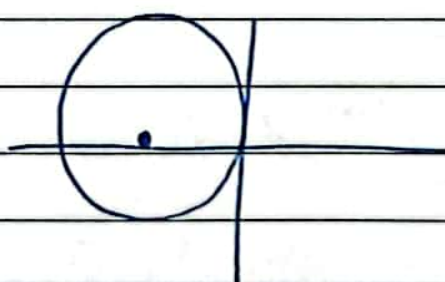
$x^2 + y^2 = a^2 \Rightarrow x = \sqrt{a^2 - y^2} \text{ or } x = -\sqrt{a^2 - y^2}$



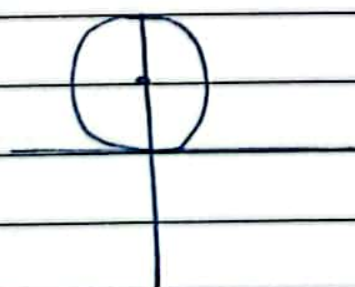
$r = 2a \cos \theta$



$r = -2a \cos \theta$



$r = 2a \sin \theta$



$r = -2a \sin \theta$

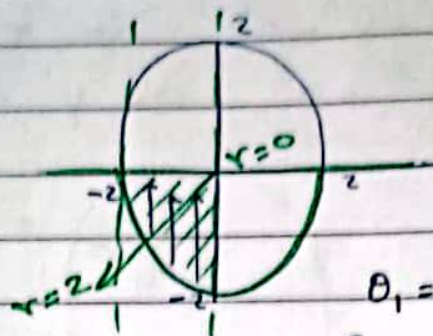


Ex. Evaluate the following iterated integral

$$\textcircled{1} \int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 e^{x^2+y^2} dy dx$$

$$y = -\sqrt{4-x^2} \rightarrow y = 0$$

$$x = -2 \rightarrow x = 0$$



$$\theta_1 = \pi / \theta_2 = \frac{3\pi}{2}$$

$$\theta_1 = -\pi / \theta_2 = -\frac{\pi}{2}$$

$$\int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 e^{x^2+y^2} dy dx = \int_{-\pi}^{-\pi/2} \int_0^2 e^{r^2} r dr d\theta =$$

$$\left(\int_{-\pi}^{-\pi/2} 1 d\theta \right) \left(\int_0^2 r e^{r^2} dr \right) = \frac{\pi(e^4 - 1)}{4}$$

$$\textcircled{2} \int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 e^{x^2+y^2} dy dx$$

$$y = -\sqrt{4-x^2} \rightarrow y = 0$$

$$x = -2 \rightarrow x = 2$$

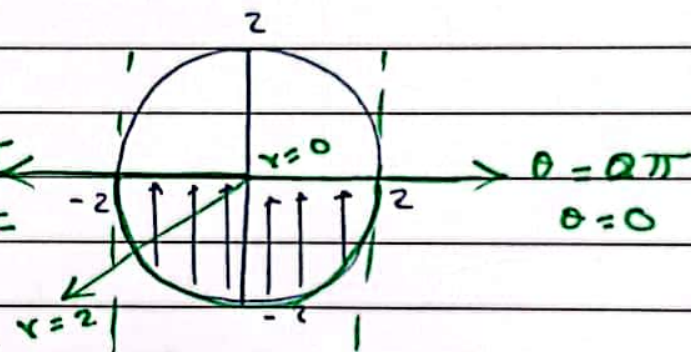
$$\theta = \pi \leftarrow$$

$$\theta = \pi -$$

$$\theta = 0 \rightarrow$$

$$\theta = 0$$

$$\int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 e^{x^2+y^2} dy dx = \int_{\pi}^{2\pi} \int_0^2 e^{r^2} r dr d\theta$$

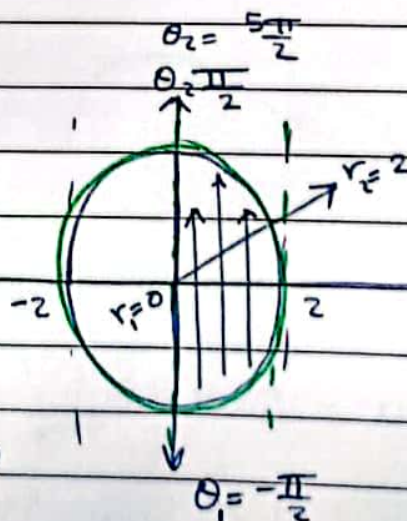


$$\textcircled{3} \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy dx$$

$$y = -\sqrt{4-x^2} \rightarrow y = \sqrt{4-x^2}$$

$$x = 0 \rightarrow x = 2$$

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy dx = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{r^2} r dr d\theta$$



$$\theta_1 = -\frac{\pi}{2}$$

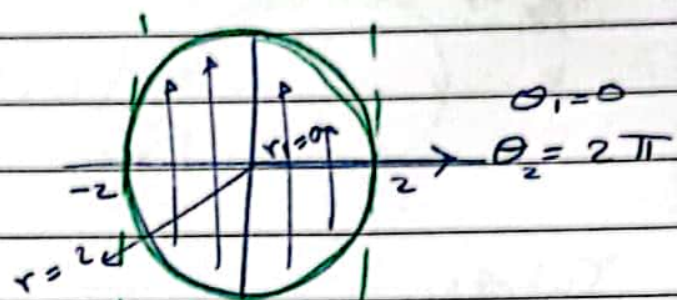
$$\theta_2 = \frac{3\pi}{2}$$

$$(4) \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy dx$$

$$y = -\sqrt{4-x^2} \rightarrow y = \sqrt{4-x^2}$$

$$x = -2 \rightarrow x = 2$$

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy dx = \int_0^{2\pi} \int_0^2 e^{r^2} r dr d\theta$$



إذا كان الـ θ والـ r + الجذر - الجذر - الجذر

$$\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = - \iint$$

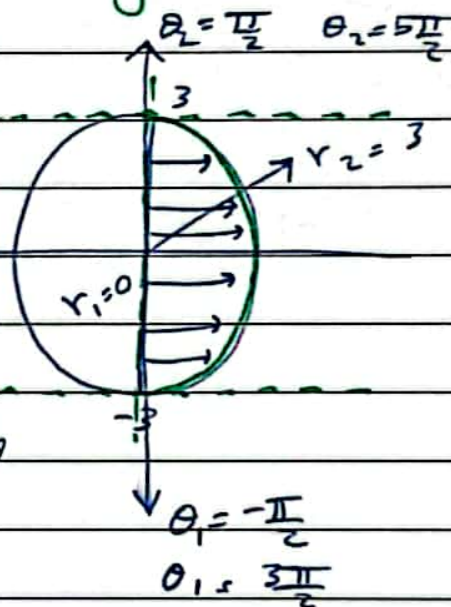
Ex. Evaluate the following iterated integrals

$$(1) \int_{-3}^3 \int_0^{\sqrt{9-y^2}} \sin(x^2+y^2) dx dy$$

$$x = 0 \rightarrow x = \sqrt{9-y^2}$$

$$y = -3 \rightarrow y = 3$$

$$\int_{-3}^3 \int_0^{\sqrt{9-y^2}} \sin(x^2+y^2) dx dy = \int_{-\pi/2}^{\pi/2} \int_0^3 r \sin r^2 dr d\theta$$

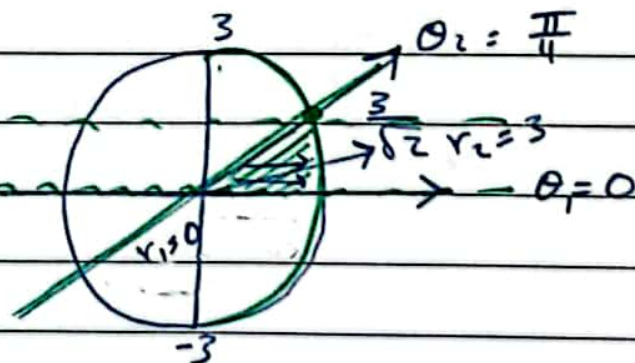


$$(2) \int_0^{\frac{3}{\sqrt{2}}} \int_y^{\sqrt{9-y^2}} \sin(x^2+y^2) dx dy$$

$$X = y \rightarrow \theta = \frac{\pi}{4} \rightarrow \frac{y}{x} = \tan \theta$$

$$y = 0 \rightarrow y = \frac{3}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$



Intersection point

$$y = \sqrt{9-y^2} \rightarrow y = \frac{3}{\sqrt{2}}$$

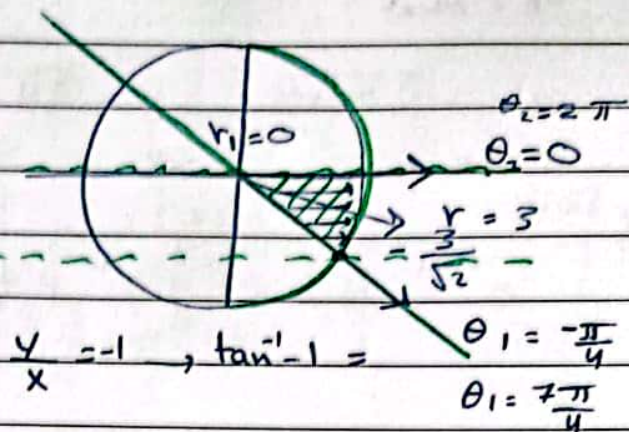
$$\int_0^{\frac{3}{\sqrt{2}}} \int_y^{\sqrt{9-y^2}} \sin(x^2+y^2) dx dy = \int_0^{\pi/4} \int_0^3 r \sin r^2 dr d\theta$$

$$\textcircled{3} \int_{-\frac{3}{\sqrt{2}}}^0 \int_{-y}^{\sqrt{9-y^2}} \sin(x^2+y^2) dx dy$$

$$x = -y \rightarrow x = \sqrt{9-y^2}$$

$$y = -\frac{3}{\sqrt{2}} \rightarrow x = 0$$

$$\text{Intersection} \rightarrow y = \pm \frac{3}{\sqrt{2}}$$



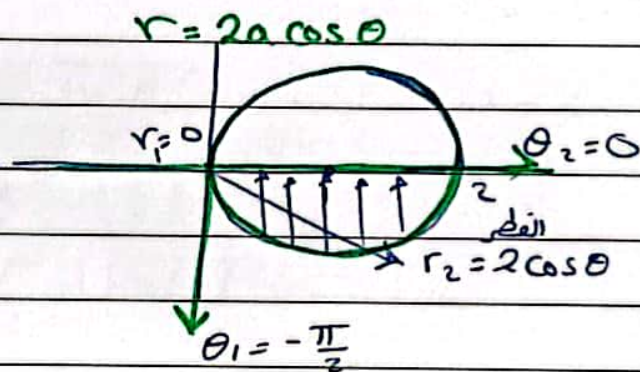
$$\int_{-\frac{3}{\sqrt{2}}}^0 \int_{-y}^{\sqrt{9-y^2}} \sin(x^2+y^2) dx dy = \int_{-\frac{\pi}{4}}^0 \int_0^3 r \sin r^2 dr d\theta$$

Ex. Evaluate the following iterated integrals

$$\textcircled{1} \int_0^2 \int_{-\sqrt{2x-x^2}}^0 \frac{1}{\sqrt{x^2+y^2}} dy dx$$

$$y = -\sqrt{2x-x^2}, y = 0$$

$$x = 0, x = 2$$



$$-\sqrt{2x-x^2} = y \rightarrow y^2 = 2x-x^2$$

$$x^2 - 2x + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

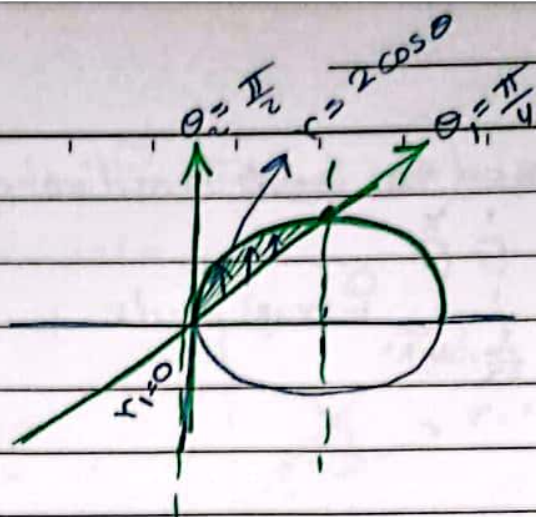
$$\int_0^2 \int_{-\sqrt{2x-x^2}}^0 \frac{1}{\sqrt{x^2+y^2}} dy dx = \int_{-\frac{\pi}{2}}^0 \int_0^{2 \cos \theta} \frac{1}{r} r dr d\theta$$

المجال هو النصف العلوي من الدائرة

$$(2) \int_0^1 \int_x^{\sqrt{2x-x^2}} \frac{1}{x \sqrt{x^2+y^2}} dy dx$$

$$y = x \rightarrow y = \sqrt{2x-x^2}$$

$$x = 0 \rightarrow x = 1$$



$$x = \sqrt{2x-x^2} \Rightarrow x^2 = 2x-x^2$$

$$2x^2 - 2x = 0$$

$$2x(x-1) = 0$$

$$x = 0, x = 1$$

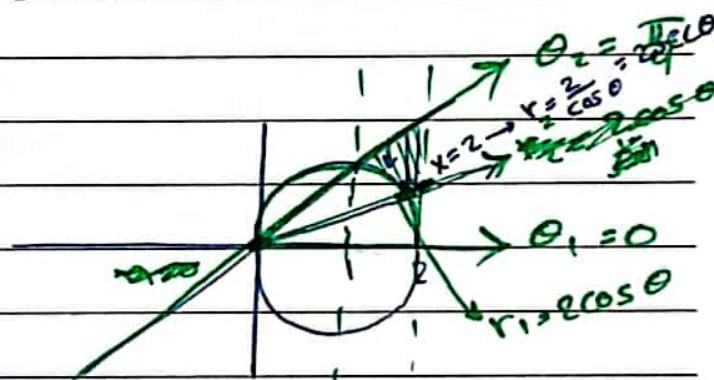
$$\int_0^1 \int_x^{\sqrt{2x-x^2}} \frac{1}{x \sqrt{x^2+y^2}} dy dx = \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} \frac{1}{r} r dr d\theta$$

$$(3) \int_1^2 \int_x^{\sqrt{2x-x^2}} \frac{1}{x \sqrt{x^2-y^2}} dy dx$$

$$y = x \rightarrow y = \sqrt{2x-x^2}$$

$$x = 1 \rightarrow x = 2$$

$$\int_1^2 \int_x^{\sqrt{2x-x^2}} \frac{1}{x \sqrt{x^2-y^2}} dy dx = \int_0^{\pi/4} \int_{2\cos\theta}^{2\sec\theta} \frac{1}{r} r dr d\theta$$

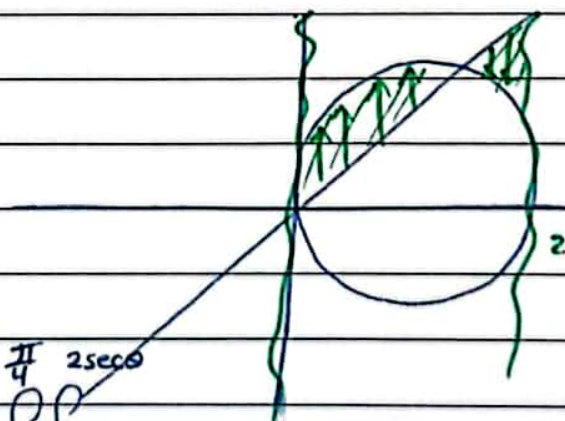


لا نه من فقه لغت يعني نه
الكبير للصغير غلط اشارة جالب

$$(4) \int_0^2 \int_x^{\sqrt{2x-x^2}} \frac{1}{x \sqrt{x^2+y^2}} dy dx$$

$$y = x \rightarrow y = \sqrt{2x-x^2}$$

$$x = 0 \rightarrow x = 2$$



$$\int_0^2 \int_x^{\sqrt{2x-x^2}} \frac{1}{x \sqrt{x^2+y^2}} dy dx = \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} \frac{1}{r} r dr d\theta + \int_0^{\pi/4} \int_{2\cos\theta}^{2\sec\theta} \frac{1}{r} r dr d\theta$$

15.7) Triple Integrals

If $P(x, y, z)$ is continuous on the rectangular box
 $B = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$
 $= [a, b] \times [c, d] \times [r, s]$ then

$$\iiint P dv = \int_a^b \int_c^d \int_r^s P dz dy dx = \int_r^s \int_c^d \int_a^b P dx dy dz = \int_a^b \int_r^s \int_c^d P dy dz dx$$

Ex. Evaluate $\iiint xyz^2 dv$ where

$$B = \{(x, y, z) : \sqrt{2} \leq x \leq 2, 0 \leq y \leq 4, -1 \leq z \leq 1\}$$

$$\iiint xyz^2 dv = \int_{-1}^1 \int_0^4 \int_{\sqrt{2}}^2 xyz^2 dx dy dz = \left(\int_{\sqrt{2}}^2 x dx \right) \left(\int_0^4 y dy \right) \left(\int_{-1}^1 z^2 dz \right)$$

Let E be the solid in 3D such that $u_1(x, y) \leq z \leq u_2(x, y)$
and the region D is the projection of S on the xy -plane

$$\text{then } \iiint P dv = \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} P dz \right) dA$$

we may take $dA = dy dx$ or $dx dy$ or $r dr d\theta$

Let E be the solid in 3D such that $u_1(y, z) \leq x \leq u_2(y, z)$

$$\text{then } \iiint P dv = \iint_D \left(\int_{u_1(y, z)}^{u_2(y, z)} P dx \right) dA$$

$dA = dy dz$ or $dz dy$ or $r dr d\theta$

Let E be the solid in 3D such that $u_1(x, z) \leq y \leq u_2(x, z)$

$$\text{then } \iiint P dv = \iint_D \left(\int_{u_1}^{u_2} P dy \right) dA$$

$dA = dx dz$ or $dz dx$ or $r dr d\theta$

Ex: Evaluate $\iiint_E z \, dV$ where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$ & $x+y+z=1$

$$dV = dz \, dA$$

المساحة المثلثية في المستوى xy هي $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$
 المساحة المثلثية في المستوى yz هي $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$
 المساحة المثلثية في المستوى xz هي $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$$z=0 \text{ \& } z=1-x-y$$

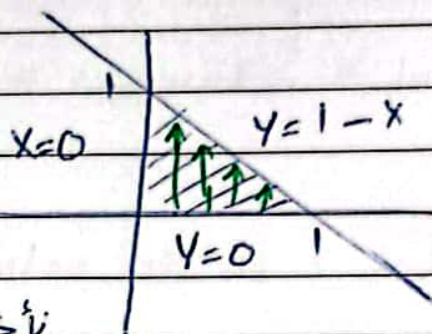
Region D bounded by $x=0$, $y=0$

intersection of surfaces

$$0 = 1 - x - y$$

$$y=0 \rightarrow y=1-x$$

$$x=0$$



$$\iiint_E z \, dV = \iint_D \left(\int_0^{1-x-y} z \, dz \right) dA$$

نأخذ
 المساحة المثلثية في المستوى xy هي $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$
 المساحة المثلثية في المستوى yz هي $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$
 المساحة المثلثية في المستوى xz هي $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \frac{1}{24}$$

Ex: Find $\iiint_E \sqrt{x^2+z^2} \, dV$ where E is the region bounded by the paraboloid $y=x^2+z^2$ & the plane $y=4$

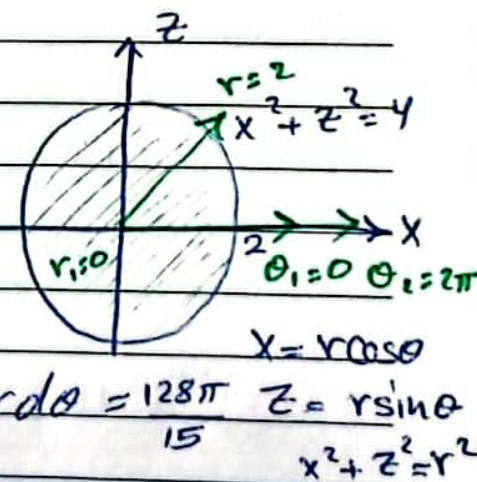
$$dV = dy \, dA$$

surfaces: $y=x^2+z^2$ & $y=4$

Region D bounded by

Intersection of surfaces $D: x^2+z^2=4$

$$\iiint_E \sqrt{x^2+z^2} \, dV = \iint_D \left(\int_{x^2+z^2}^4 \sqrt{x^2+z^2} \, dy \right) dA =$$



$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dy \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^2 (4-r^2) \, dr \, d\theta = \frac{128\pi}{15}$$

$x = r \cos \theta$
 $z = r \sin \theta$
 $x^2 + z^2 = r^2$

Ex: let $I = \int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) dz dy dx$

1) Express the iterated integral I as a triple integral

Equations: $z=0, z=y$ / $y=0, y=x^2$ / $x=0, x=1$
 Surfaces Curves
 $\int_0^1 \int_0^{x^2} \int_0^y$

intersection $\rightarrow z=0, z=y$ / $y=0, y=x^2$ / $x=0, x=1$
 $y=0$ $x=0$ $x=1$

Let E be the solid bounded by: $z=0, z=y, y=x^2, x=1$

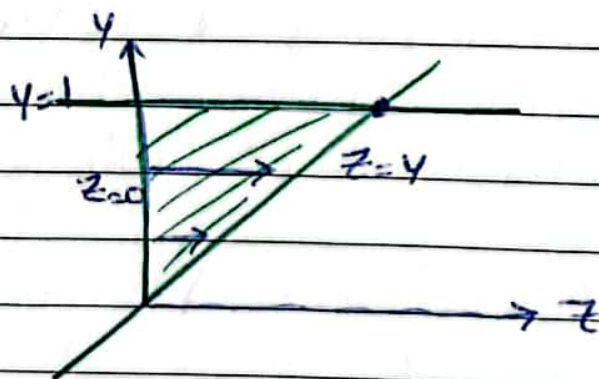
$I = \iiint_E f dV$

2) Rewrite the iterated integral I integrating first with respect to x , then z , and then y

$dV = dx dz dy \rightarrow x$

$x=1, x=\sqrt{y} \Rightarrow 0 \leq x \leq 1 \Rightarrow$ surfaces
 Region $D: z=0, z=y$

Intersection of surfaces



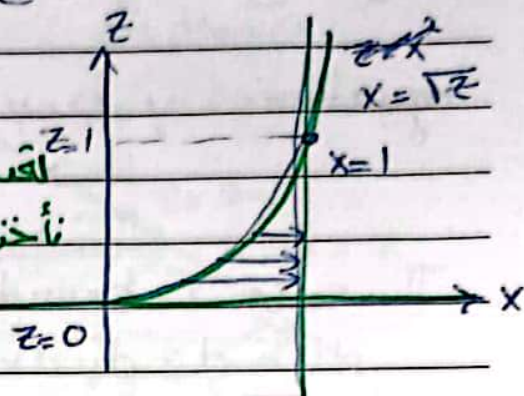
$\sqrt{y} = 1 \rightarrow y=1$

$I = \iiint_E f dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x, y, z) dx dz dy$

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③ Rewrite the iterated integral I integrating first with respect to y then x & then z من الأسفل إلى الأعلى
 $dV = dy dx dz$

surfaces : $y = z$, $y = x^2$ أعلى من الأسفل
 Region D : $z = 0$, $x = 1$ نأخذ نقطة اختيارية
أبواب



intersection of surfaces

$$z = x^2 \Rightarrow z = x = \sqrt{z} \Rightarrow 0 \leq x \leq 1 \Rightarrow x = \sqrt{z}$$

$$I = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} F(x, y, z) dy dx dz$$

Ex: Rewrite the iterated integral $\int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} \int_0^{2-y} F(x, y, z) dz dy dx$ integrating first with respect to x, y, z

$$z = 0, z = 2 - y \mid y = \frac{x}{2}, y = \sqrt{x} \mid x = 0, x = 4$$

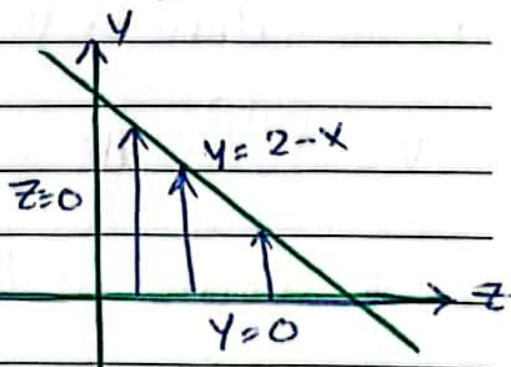
من الأسفل إلى الأعلى $y = 2x$ من الأسفل إلى الأعلى $x = 0, x = 4$

$$dV = dx dy dz$$

The solid E bounded by $z = 0$, $z = 2 - y$, $y = \frac{x}{2}$, $y = \sqrt{x}$

surfaces : $x = 2y$, $x = y^2$
 Region D : $z = 0$, $z = 2 - y$

Intersection of surfaces
 $y^2 = 2y \Rightarrow y = 0, y = 2$



$$\int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} \int_0^{2-y} F dz dy dx = \int_0^2 \int_0^{2-z} \int_{y^2}^{2y} F dx dy dz$$

Ex: Express the iterated integral $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f \, dy \, dz \, dx$
 integrating first respect to z then y and then x .

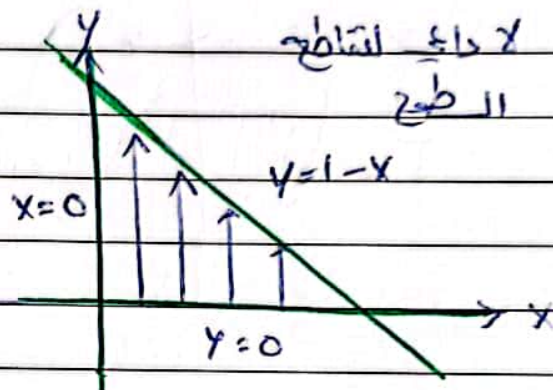
equations: $y=0, y=1-x$ / $z=0, z=1-x^2$ / $x=0, x=1$

The solid E bounded by $y=0, y=1-x, z=0, z=1-x^2, x=0$
 $dV = dz \, dy \, dx$

surfaces: $z=0, z=1-x^2$

Region D : $y=0, y=1-x, x=0$

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f \, dy \, dz \, dx = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f \, dz \, dy \, dx$$



volumes

Let E be the solid in 3D such that $u_1(x,y) \leq z \leq u_2(x,y)$
 and the region D is the projection of S on the xy -plane

1) The volume of the solid E can be expressed as triple integral

$$V = \iiint 1 \, dV = \iint \left(\int_{u_1}^{u_2} 1 \, dz \right) dA$$

2) The volume of the solid E can be expressed as double integral

$$V = \iint (u_2 - u_1) \, dA$$

Ex: Use triple integral to find the volume of the solid enclosed by the parabolic cylinder $y = z^2$ & the planes $y = x$, $z = 2 - x$

surfaces: $y = z^2$, $y = x$

$$dV = dy dz dx$$

Region D: $z = 2 - x$

Intersection of surfaces

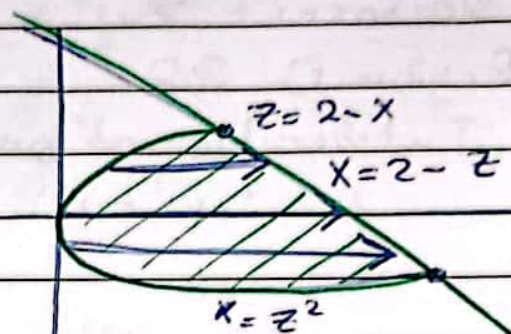
$$y = z^2, y = x$$

$$z = 2 - x, z^2 = x$$

$$dA = dx dz$$

Intersection of curves $\Rightarrow z = -2, z = 1$

$$V = \iiint 1 dV = \iint \left(\int_{z^2}^{2-z} 1 dy \right) dA = \int_{-2}^1 \int_{z^2}^{2-z} 1 dx dz$$



Ex: Use triple integral to find the volume of the tetrahedron T enclosed by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, $z = 0$

$$dV = dz dy dx$$

surfaces: $z = 0$, $z = 2 - x - 2y$

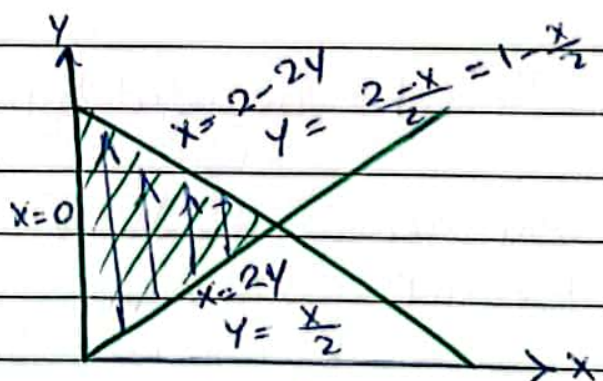
Region D: $x = 2y$, $x = 0$

Intersection of surfaces

$$2 - x - 2y = 0$$

$$x = 2 - 2y$$

$$V = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} 1 dz dy dx$$



$$dA = dy dx$$

Intersection of lines

$$y = \frac{1}{2} \Rightarrow x = 1$$

Ex. 1

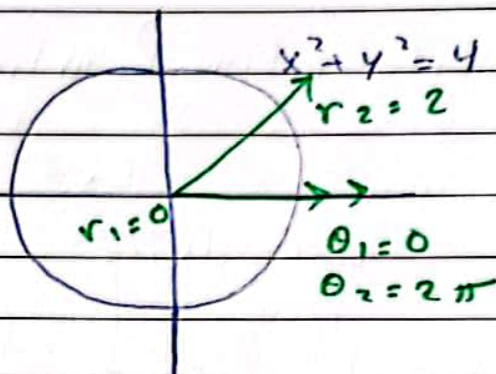
Ex. Set up as a double integral the volume of the solid bounded by $z=1$, $z=5-x^2-y^2$

surfaces: $z=1$, $z=5-x^2-y^2$

Region D ??

Intersection of surfaces

$$5-x^2-y^2=1 \rightarrow x^2+y^2=4$$



$$V = \iint_D (5-x^2-y^2-1) dA =$$

$$\iint_D (4-(x^2+y^2)) dA =$$

$$\int_0^{2\pi} \int_0^2 (4-r^2) r dr d\theta$$

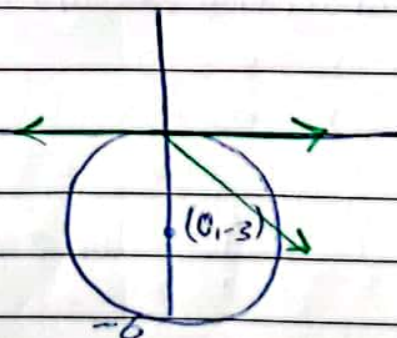
Ex. Set up as a triple integral the volume of the solid subl lies (under the cone $z=\sqrt{x^2+y^2}$) & (above the xy -plane) and inside the cylinder $x^2+y^2=-6y$ $z=0$ do it

The solid bounded by $z=\sqrt{x^2+y^2}$, $z=0$, $x^2+y^2=-6y$

surfaces: $z=\sqrt{x^2+y^2}$, $z=0$

Region D: $x^2+y^2=-6y$

$$x^2+(y+3)^2=9$$



$$V = \iiint_D 1 dV = \iint_D \int_0^{\sqrt{x^2+y^2}} 1 dz dA$$

$$= \int_{-\pi}^{\pi} \int_0^{-6 \sin \theta} \int_0^r r dz dr d\theta$$

15.8) Triple Integral in Cylindrical Coordinates

let $A(x, y, z)$ be a pt in rectangular (Cartesian) coordinates
 The cylindrical coordinates of A are $A(r, \theta, z)$ where
 $r = \sqrt{x^2 + y^2}$, $\tan \theta = \frac{y}{x}$, $x = r \cos \theta$, $y = r \sin \theta$

Ex: Plot the pt with cylindrical coordinates $A(2, \frac{2\pi}{3}, 1)$
 and find its rectangular coordinates

$$A(2, \frac{2\pi}{3}, 1) \Rightarrow r=2, \theta = \frac{2\pi}{3}, z=1$$

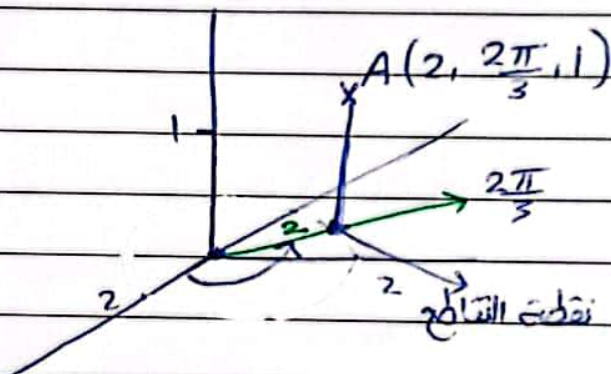
$$\sqrt{x^2 + y^2} = 2$$

$$(1, 1) \quad r=2$$

$$x = r \cos \theta = 2 \cos \frac{2\pi}{3} = -1$$

$$y = r \sin \theta = 2 \sin \frac{2\pi}{3} = \sqrt{3}$$

$$A(-1, \sqrt{3}, 1)$$



z Sol

Ex: Find the (cylindrical coordinate) of the pts whose
 (cartesian) coordinate are $A(3, -3, 7)$ $B(-3, 3, 7)$ $C(-3, -3, 1)$
 rectangular $D(2, 0, -1)$ $E(-2, 0, 1)$ $F(0, 2, -1)$
 $G(0, -2, 1)$

Ex: $A(3, -3, 7) \Rightarrow r = \sqrt{x^2 + y^2} = 3\sqrt{2}$ / $\tan \theta = \frac{y}{x} \Rightarrow \theta = \frac{7\pi}{4}, -\frac{\pi}{4}$
 $A(3\sqrt{2}, \frac{7\pi}{4}, 7)$ or $(3\sqrt{2}, \frac{7\pi}{4}, 7)^x$

$$B(-3, 3, 7) \Rightarrow$$

Ex: Describe & sketch whose eq in cylindrical coordinates is

1) $z=r$ 2) $z=-r$ 3) $z=\sqrt{6-r^2}$

1) $z=r \rightarrow z=\sqrt{x^2+y^2}$ cone

2) $z=-r \rightarrow z=-\sqrt{x^2+y^2}$ cone

3) $z=\sqrt{6-r^2} \rightarrow r^2+z^2=6 \rightarrow x^2+y^2+z^2=6$, hemisphere

Ex: Write the eq $z=x^2-y^2$ in cylindrical coordinates

$$z = r^2 \cos^2 \theta - r^2 \sin^2 \theta = r^2 (\cos^2 \theta - \sin^2 \theta) = r^2 \cos 2\theta$$

Ex: Identify the surface & write its eq in Cartesian Coordinates

1) $r=5$ 2) $z=4-r^2$ 3) $\theta=\frac{\pi}{4}$ 4) $\theta=0$ 5) $\theta=\frac{\pi}{2}$ 6) $\theta=\pi$ 7) $\theta=\frac{2\pi}{3}$

1) $r=5 \Rightarrow x^2+y^2=25 \Rightarrow$ cylinder

- 2) $z = 4 - r^2 \Rightarrow z = 4 - (x^2 + y^2)$ paraboloid
 3) $\tan \theta = \tan \frac{\pi}{4} \Rightarrow \frac{y}{x} = 1 \Rightarrow x = y$ plane
 4) $\tan \theta = \tan 0 \Rightarrow \frac{y}{x} = 0 \Rightarrow y = 0$ xz -plane $x \geq 0$
 5) $\tan \theta = \tan \frac{\pi}{2} \Rightarrow \frac{y}{x} = \infty$ half yz -plane, $y \geq 0$
 6) $\tan \theta = \tan \pi \Rightarrow \frac{y}{x} = 0 \Rightarrow y = 0, x \leq 0$ half xz -plane
 7) $\tan \theta = \tan \frac{3\pi}{2} \Rightarrow \frac{y}{x} = -\infty \Rightarrow x = 0, y \leq 0$ half yz -plane

Let $F(x, y, z)$ be cont on the solid S in $\mathbb{R}^3$ whose lower surface is $z = g_1(x, y)$ & upper surface $z = g_2(x, y)$ and its projection on the xy -plane is the region

$$D = \{(r, \theta) \mid h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta\}$$

where $0 \leq \beta - \alpha < 2\pi$ then

$$\iiint F dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1}^{g_2} F(r \cos \theta, r \sin \theta, z) dz r dr d\theta$$

Ex: Evaluate $T = \iiint (x + y + z) dV$ where F is the solid in the first octant that lies under the paraboloid $(z = 12 - 3x^2 - 3y^2)$ upper

surfaces: $z = 12 - 3x^2 - 3y^2$; $z = 0$

Region D : $x = 0, y = 0$

First octant

$x \geq 0 / y \geq 0 / z \geq 0$

Intersection surfaces

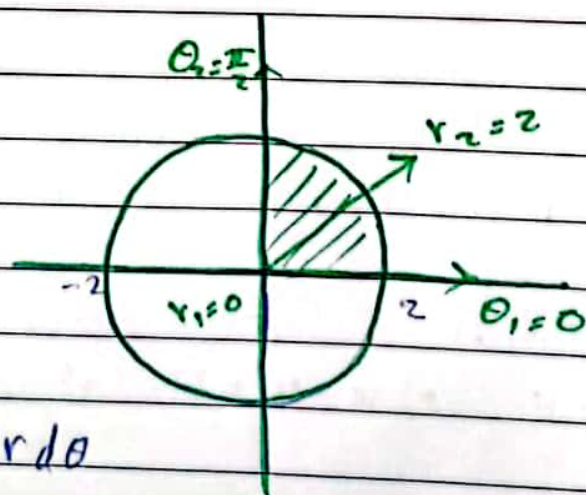
$$12 - 3x^2 - 3y^2 = 0$$

$$x^2 + y^2 = 4$$

$$12 - 3x^2 - 3y^2$$

$$T = \iiint (x + y + z) dz r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{12-3r^2} (r \cos \theta + r \sin \theta) r dz dr d\theta$$



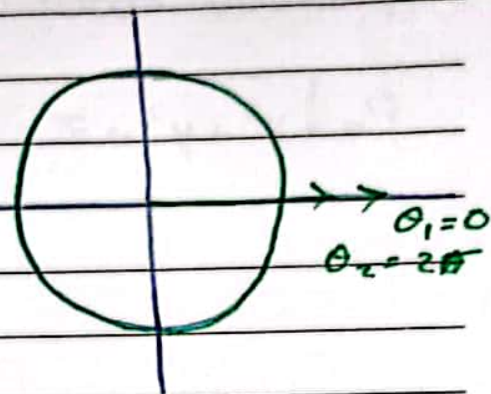
Ex: Find the volume of the solid S that lies (within) the cylinder $x^2 + y^2 = 1$ (below) the plane $z = 1$ & (above) the paraboloid $z = 1 - x^2 - y^2$ using triple integrals

surfaces: $z = 1$, $z = 1 - x^2 - y^2$

Region D : $x^2 + y^2 = 1$

$$V = \iiint_D 1 \, dV = \iiint_{1-x^2-y^2}^1 dz \, dA$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^1 r \, dz \, dr \, d\theta$$



Ex: Evaluate ① $I_1 = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx$

$z = \sqrt{x^2 + y^2}$, $z = 2$ / $y = -\sqrt{4-x^2}$, $y = \sqrt{4-x^2}$ / $x = -2$, $x = 2$
 $y = \pm \sqrt{4-x^2}$

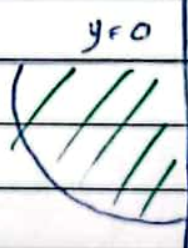
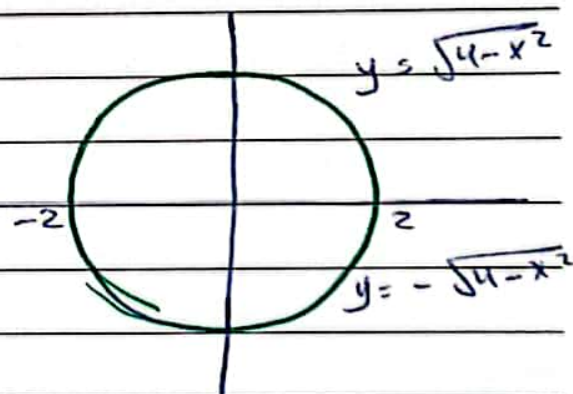
$$I_1 = \int_0^{2\pi} \int_0^2 \int_r^2 r^3 \, dz \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_0^2 r^5 (2-r) \, dr \, d\theta = \frac{16\pi}{5}$$

② $I_2 = \int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx$

Region D : $y = -\sqrt{4-x^2}$ → $y = 0$
 $-2 \leq x \leq 0$

$$I_2 = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^2 \int_r^2 r^3 \, dz \, dr \, d\theta$$



15.9) Triple Integrals in Spherical Coordinates

Let $A(x, y, z)$ be a pt in rectangular coordinates. The spherical coordinates of A are (ρ, θ, ϕ) where

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad , \quad \begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$

$$r = \rho \sin \phi \quad \rho = \sqrt{r^2 + z^2} \quad \cos \phi = \frac{z}{\rho}$$

$$0 \leq \theta \leq 2\pi \quad / \quad 0 \leq \phi \leq \pi$$

Ex: The pt $A(2, \frac{\pi}{4}, \frac{\pi}{3})$ is given in spherical coordinates plot the pt showing its spherical coordinates and find its rectangular & cylindrical coordinates

CH 12) solutions → 1316 ✨ تلخيص قوانين المادة

$$\left. \begin{array}{l} xy\text{-plane} \rightarrow z=0 \\ xz\text{-plane} \rightarrow y=0 \\ yz\text{-plane} \rightarrow x=0 \end{array} \right\} \text{coordinate (planes)}$$

$$\left. \begin{array}{l} x\text{-axis} \rightarrow y=0, z=0, x=t \\ y\text{-axis} \rightarrow x=0, z=0, y=t \\ z\text{-axis} \rightarrow x=0, y=0, z=t \end{array} \right\} \text{coordinate (axis)}$$

let $A(a, b, c)$ Find:

The distance between A & xy -plane $\Rightarrow |z| = c$

The distance between A & xz -plane $\Rightarrow |y| = b$

The distance between A & yz -plane $\Rightarrow |x| = a$

The distance between A & x -axis $\Rightarrow \sqrt{b^2 + c^2}$

The distance between A & y -axis $\Rightarrow \sqrt{a^2 + c^2}$

The distance between A & z -axis $\Rightarrow \sqrt{b^2 + a^2}$

$$A(a, b, c) \cdot \overset{F}{\underset{x}{\text{---}}} \cdot B(d, e, f)$$

$$\text{The midpoint} = \left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2} \right)$$

$$\text{The distance between } A \& B = \sqrt{(d-a)^2 + (e-b)^2 + (f-c)^2}$$

The equation of sphere of radius r & center (a, b, c)

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

largest sphere } z محال = y محال = x محال ① $a=b=c$
 ② r محال = x محال = y محال = z محال ③

point (a, b, c) is the center of the sphere. r is the radius.

Vectors

$A(a, b, c)$ $B(d, e, f)$

$$\vec{AB} = \langle d-a, e-b, f-c \rangle$$

$$|\vec{AB}| = \sqrt{(d-a)^2 + (e-b)^2 + (f-c)^2}$$

A vector $\vec{v}$ is called a unit vector if $|\vec{v}| = 1$

Dot Product

$A(a, b, c)$ $B(d, e, f)$

$$\vec{A} \cdot \vec{B} = a \times d + b \times e + c \times f = \text{scalar}$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$\vec{u} \perp \vec{v} \Rightarrow \vec{u} \cdot \vec{v} = 0$$

$\vec{u} \cdot \vec{v} \geq 0 \rightarrow \theta$ is a acute angle

$\vec{u} \cdot \vec{v} < 0 \rightarrow \theta$ is an obtuse angle

$\vec{u} \cdot \vec{v} = 0 \rightarrow \theta$ is a right angle

scalar projection of $\vec{u}$ onto the $\vec{v}$ is

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

vector projection of $\vec{u}$ onto the $\vec{v}$ is

$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$

$\Rightarrow \theta = 0 \rightarrow \vec{u} \& \vec{v}$ have the same direction

$\Rightarrow \theta = 180 \rightarrow \text{proj}_{\vec{v}} \vec{u} = -\vec{u}$

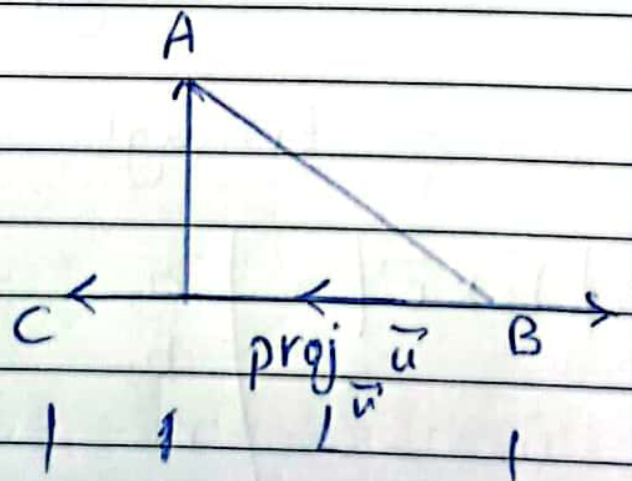
$\Rightarrow \theta = 90 \rightarrow \text{proj}_{\vec{v}} \vec{u} = 0$

$$* \text{comp}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta$$

$$* \text{proj}_{\vec{v}} \vec{u} = \left| \text{comp}_{\vec{v}} \vec{u} \right| \frac{\vec{v}}{|\vec{v}|}$$

To Find the Dis ($A \rightarrow L$)

$$|\vec{u} - \text{proj}_{\vec{v}} \vec{u}|$$



Cross product

$$\begin{aligned}\vec{a} &= \langle a_1, a_2, a_3 \rangle & \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ \vec{b} &= \langle b_1, b_2, b_3 \rangle\end{aligned}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

The length of $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$

$$|\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}$$

the distance from A to line l = $\frac{|\vec{u} \times \vec{v}|}{|\vec{u}|}$
where $\vec{u} = \vec{BA}$, $\vec{v} = \vec{BC}$

two vectors $\vec{a}$ & $\vec{b}$ are parallel if

$$\vec{a} \times \vec{b} = 0 \quad \&$$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

points are collinear if $\vec{AB} \parallel \vec{AC}$

The area of parallelogram = $|\vec{a} \times \vec{b}|$

The area of triangle = $\frac{1}{2} |\vec{a} \times \vec{b}|$

$$\vec{a} \cdot |\vec{b} \times \vec{c}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

⇒ Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ لا تملك ثلاثة متجهات

⇒ 4 points A, B, C, D are coplanar if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ where $\vec{a} = \vec{AB}, \vec{b} = \vec{AC}, \vec{c} = \vec{AD}$

The volume of the parallelepiped = $|\vec{a} \cdot (\vec{b} \times \vec{c})|$

equations of lines & planes

parametric eq. ⇒ $x = x_0 + at / y = y_0 + bt / z = z_0 + ct$

symmetric eq ⇒ $\frac{x-x_0}{a} / \frac{y-y_0}{b} / \frac{z-z_0}{c}$ ينقسم من اوايه

vector eq ⇒ $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$
 $= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

The equation of the plane $P(x_0, y_0, z_0)$ & with normal vector $\vec{n} = \langle a, b, c \rangle$

eq ⇒ $ax + by + cz = ax_0 + by_0 + cz_0$

معادلات التماسح لكونه مع المعطيات

إذا كان معطى متوى بالخط W تقاطع وتريد متجه عمودي على W فوجد الاتجاهات cross product

ما لكونه في متوالية وحظ تقاطع لهم ① فوجد نقطة على خط التقاطع وذلك بتعويض اي قيم x, y, z

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1| |n_2|}$$

$$\text{Distance} = \frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}}$$

point

$$ax + by + cz = 0$$

← $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$

$P_1 \neq P_2 \Rightarrow$ the distance between P_1 & $P_2 = 0$
 $P_1 = P_2 \Rightarrow$ " " " "

① $ax + by = c \rightarrow$ line

$2x - 3y = 5 \rightarrow$ line in xy -plane

$z + 6x = 4 \rightarrow$ line in xz -plane

② $ax^2 + by = c \rightarrow$ parabola

$3x^2 - y = 6 \rightarrow$ parabola in xy -plane

$y^2 + 3z = 0 \rightarrow$ parabola in yz -plane

③ $x^2 + y^2 = r^2 \rightarrow$ circle

$(x-1)^2 + y^2 = 5 \rightarrow$ circle in xy -plane

④ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow$ ellipse $a \neq b$

⑤ $\frac{x^2}{a} - \frac{y^2}{b} = 1 \rightarrow$ hyperbola in plane

$x^2 - y^2 = 5$

⑥ ellipsoid $\rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

⑦ Cone $\rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

⑧ elliptic parabola $\rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

CH 13:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\text{Dom}(t) \text{ Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$$\text{If } \lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$$

$$\vec{r}(t) \text{ is cont. when } \lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$$

tangent vector $\rightarrow r'(t)$

unit tangent vector